

Colored knot polynomials. HOMFLY in representation $[2, 1]$

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Abstract

This paper starts a systematic description of colored knot polynomials, beginning from the first non-(anti)symmetric representation $R = [2, 1]$. The project involves several steps:

- (i) parametrization of big families of knots *a la* [1],
- (ii) evaluating Racah/mixing matrices for various numbers of strands in various representations *a la* [2],
- (iii) tabulating and collecting the results at [3].

In this paper we discuss only representation $R = [2, 1]$ and construct all necessary ingredients that allow one to evaluate knot/links represented by three strand closed parallel braids with inserted double-fat fingers. In particular, it is used to evaluate knots from a 7-parametric family: this family contains over 80% of knots with up to 10 intersections, but does not include mutants.

1 Introduction

Knot polynomials [4] are among the hottest subjects of modern theory, interesting both for physics and mathematics. This is because they are correlators of Wilson lines in 3d Chern-Simons (CS) theory [5, 6] (sometimes deformed) and are naturally related to 2d conformal blocks, both with extended symmetry (WZWN models) and without it, [6, 7, 8, 9, 10], to matrix models and tau-functions [11, 12, 14, 15, 13] – the main types of special functions we currently know and use. Therefore, it is not a surprise that knot polynomials appear in quantitative description of various seemingly unrelated problems, from augmentation varieties in Calabi-Yau manifolds [16] to the theory of quantum Hall effect. They satisfy a vast variety of linear and non-linear relations, of which we yet discovered only a small fraction [17, 18], and this makes us to believe that complete theory of knot polynomials is actually available.

The most important HOMFLY polynomials

$$H_R^{\mathcal{L} \subset \mathcal{M}}(q, A) = \left\langle \text{Tr}_R P \exp \oint_{\mathcal{L}} A \right\rangle^{CS} \quad (1)$$

depend on five types of arguments: the knot/link \mathcal{L} in a three-dimensional space \mathcal{M} , the CS coupling constant $q = \exp\left(\frac{2\pi i}{k+N}\right)$, the Lie algebra $G = SU(N)$ with $A = q^N$ and its representation (Young diagram) R . Similar polynomials can be defined for other Lie algebras and groups, they are usually related to HOMFLY polynomials

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[19]. These quantities are called polynomials, because, being properly normalized they are Laurent polynomials in variables q and A , at least when \mathcal{M} is simply-connected: $\mathcal{M} = R^3$ or S^3 . Polynomiality is preserved for virtual knots [20, 21].

Today, of most interest is taming the dependencies of HOMFLY polynomials on \mathcal{L} and R . For this purpose, a number of methods was developed to describe their dependencies on A and q for given \mathcal{L} and R , and the goal of this and the subsequent papers in this series is to find and list these dependencies at [3].

1.1 Modern versions of RT formalism

The most effective approach to evaluate HOMFLY polynomials is based on various versions of the Reshetikhin-Turaev (RT) approach [22],[7],[2],[23]-[28],[1]. Its starting point is substitution of the link by its link diagram, a (2,2)-valent oriented graph on a plane with two types of vertices (black and white), which is planar for ordinary links and knots (but is non-planar for virtual ones). After that, one substitutes the black and white vertices with the quantum \mathcal{R} -matrix of the quantum group $SU_q(N)$ and the inverse \mathcal{R} -matrix respectively and contracts the indices, with additional weights q^ρ per each Seifert cycle (this procedure requires a seemingly small, but not fully understood modification for virtual knots, where currently the only applicable is a very different hypercube method of [29, 30, 21]). Despite the universal \mathcal{R} -matrices are long available, their convolutions and traces in the concrete representation are not so easy to evaluate and one needs additional tricks to do this in a computationally effective way.

The key fact is that in each irreducible representation Q in the product

$$R_1 \otimes R_2 = \oplus_Q W_Q \otimes Q \quad (2)$$

the quantum \mathcal{R} -matrix acts proportionally to the identity operator, with eigenvalues $\epsilon_Q q^{\kappa_Q}$ where $\kappa_Q = \sum_{(i,j) \in Q} (i - j)$ and the sign factor ϵ_Q is plus or minus. For $R_1 = R_2$ it is plus or minus depending on whether Q belongs to symmetric and antisymmetric square of R respectively. Note that because of these sign factors Q can enter (2) twice, as Q_+ and Q_- , and the intertwiner spaces W_{Q_+} and W_{Q_-} are *not* unified.

As a corollary of this general feature in the particular case of the fundamental representation, $R_1 = R_2 = [1] = \square$, the \mathcal{R} -matrix satisfies the quadratic equation

$$(\mathcal{R} - q)(q\mathcal{R} + 1) = 0 \iff \mathcal{R} - \mathcal{R}^{-1} = q - q^{-1} \quad (3)$$

Changing normalization of the \mathcal{R} -matrix¹, one arrives at the skein relation [31],

$$(A\mathcal{R}_t - q)(Aq\mathcal{R}_t + 1) = 0 \iff A\mathcal{R}_t - A^{-1}\mathcal{R}_t^{-1} = q - q^{-1} \quad (4)$$

which is very effective for evaluating the HOMFLY polynomials in the fundamental representation, however a more sophisticated *cabling* procedure is required in other representations, which becomes effective [26] only in combination with other insights.

As soon as the \mathcal{R} -matrix acts trivially on the whole irreducible representation, one can consider it as acting on the space of intertwining operators. From now on, we keep the same notation \mathcal{R} for the \mathcal{R} -matrices acting on these spaces. These \mathcal{R} -matrices can act non-trivially only when these spaces are not one-dimensional.

The next simplification occurs when one considers fragments of link diagrams, which are braids. Acting on the m -strand braid are $m - 1$ different \mathcal{R} -matrices: $\mathcal{R}_{i,i+1}$ stands at the intersection of the strands i and $i + 1$. If one diagonalizes $\mathcal{R}_{1,2}$, then $\mathcal{R}_{i,i+1} = \mathcal{U}_i \mathcal{R}_{1,2} \mathcal{U}_i^{-1}$ and \mathcal{U}_i are called mixing matrices [2]. Clearly, $\mathcal{U}_1 = Id$, and \mathcal{U}_2 are just the Racah matrices, while higher \mathcal{U}_i are complicated convolutions of various Racah matrices (see [9] for some examples). The only case when an explicit description of all $\mathcal{R}_{i,i+1}$ is currently available, is the fundamental representation, when they are represented by beautiful sums over paths in the representation tree [25], which provides a nice competitive alternative to the skein-relation technique. Moreover, cabling is far more effective, if based on this formalism [26], still it remains a somewhat artificial procedure to describe non-fundamental representations. It would be very desirable to find a more straightforward representation theory approach to the problem.

As an example, in [24] (see also [43]), it was conjectured that in general the entries of \mathcal{U}_i are actually made from the eigenvalues of the corresponding \mathcal{R} . We partly use this conjecture in sec.3.3 below.

¹In fact, this changing of normalization leads to the HOMFLY polynomials in the topological framing, while the normalization that comes from the universal \mathcal{R} -matrix and is consistent with the group theory structure corresponds to the vertical framing.

1.2 The present paper

The main task of the present paper is to start a systematic investigation of the Racah matrices in different representations. We do this by a brute force method of [2], just evaluating the highest weights of representations Q , made from those of representations R_i in two ways:

$$(R_1 \otimes R_2) \otimes R_3 = \oplus_Q \mathcal{W}_Q \otimes Q \quad (5)$$

and

$$R_1 \otimes (R_2 \otimes R_3) = \oplus_Q \mathcal{W}'_Q \otimes Q \quad (6)$$

and rotating one set of the highest weights for the given Q into another. In this particular paper we concentrate on the case of $R_1 = R_2 = R_3 = [2, 1]$, the simplest one where the Racah matrices are unknown.

In a recent breakthrough paper [32] slightly different matrices were found, when either R_2 or R_3 are conjugate $[2, 1]$ (they are named respectively \bar{S} and S in [28] and [1]), so that the matrices essentially depend on N , – but instead only the contributions with $Q = [2, 1]$ were picked up at the r.h.s., which is a great simplification as compared to what we do in the present text.

While the results of [32] were sufficient to consider highly non-trivial examples of $[2, 1]$ -colored HOMFLY in [27] and [28], including some mutants, mixing this with the knowledge of arbitrary Racah matrices in the inclusive channel $[2, 1]^3 \rightarrow all$ allows one to do the same for entire families from [1], which are targeted at (and almost capable to) exhaustively describing all knots with restricted number of intersections. Of course, there is no way to present such a wide set of data in a paper, these concrete results will be gradually posted at [3], which is supposed to complement [37] and [38] by lists of colored polynomials and their differential expansions in the style of [35, 36].

1.3 Knot/link families

There is still no systematic classification of knots and links, different from a somewhat arbitrary enumeration in the Rolfsen tables [37, 38]. In [3], we also begin to use classification by families, suggested in [35, 33, 28] and [1].

The old idea is that every knot and link has a closed braid representation, thus one can study knot polynomials as functions of the number of strands and braid words, what can be effectively done by the evolution method [39, 35, 40, 33]. Technical problem here is that the minimal needed number of strands can be pretty big for rather simple knots (like twist knots with many twists). This is also a sign of the bad correlation between the minimal intersection number in the link diagram and the minimal number of strands in the braid representation.

Another idea [41, 42, 28] is to represent the knots and links as the two-bridge "fingers" and "propagators" [28]. These two ideas were recently combined [1]: the two-bridge "fingers" and "propagators" can be attached to closed braids with a low number of strands. Already two [28] and three [1] strands provide very big families. This looks like a far-going generalization of the pretzel family, studied in [40, 33, 9], and is equally effective: it should be possible to provide the answers for generic (anti)symmetric representations, for $R = [2, 1]$ and, as long as the corresponding Racah matrices are built in the next papers of the present series, for more sophisticated representations R . In this kind of formulas, the knots/links are parameterized by braid words, where numbers are substituted by the finger data (which are the braid words themselves). Depending on relevance/need of the "propagators", the parametrization can become even more interesting.

Of course, what is enumerated in this way, are *not* the *primary* knots and links: there are many composite knots/links and the same knot/link appears many times. New kind of questions appear, like the abundance of a given knot in a given sub-family, and the search of the most "adequate" constituents (fingers?) for a given knot. This can finally help to make the RT formalism really *local* so that it would represent complicated primary knots as being composed from the "elementary blocks", and finally find a classification based on the complexity of these blocks, while at the present stage we can distinguish "complicated" knots only visually and intuitively. This is a long way to go, still we feel that time is coming to proceed to this kind of analysis, and the data in [3] will be grouped to allow such attempts.

2 On the highest weight calculus of ref.[2]

The Racah matrix U is a transformation matrix from one orthonormal basis (I) to another (II), which are defined as follows:

$$(R \otimes R) \otimes R \xrightarrow{I} Q \xrightarrow{U} R \otimes (R \otimes R) \xrightarrow{II} Q \quad (7)$$

In our case $R = [2, 1]$ and Q is arbitrary, but most of them are equal to zero except for finitely many terms. To find nonzero terms, one can use the Littlewood-Richardson rule:

$$\chi_{R_1} \cdot \chi_{R_2} = \sum_Q C_{R_1, R_2}^Q \cdot \chi_Q \quad (8)$$

$$\begin{aligned} \chi_{[2,1]}^3 &= \chi_{[6,3]} + 2\chi_{[6,2,1]} + \chi_{[6,1,1,1]} + 2\chi_{[5,4]} + 6\chi_{[5,3,1]} + 4\chi_{[5,2,2]} + 6\chi_{[5,2,1,1]} + 2\chi_{[5,1,1,1,1]} + 4\chi_{[4,4,1]} + \\ &+ 8\chi_{[4,3,2]} + 9\chi_{[4,3,1,1]} + 9\chi_{[4,2,2,1]} + 6\chi_{[4,2,1,1,1]} + \chi_{[4,1,1,1,1,1]} + 2\chi_{[3,3,3]} + 8\chi_{[3,3,2,1]} + \\ &+ 4\chi_{[3,3,1,1,1]} + 4\chi_{[3,2,2,2]} + 6\chi_{[3,2,2,1,1]} + 2\chi_{[3,2,1,1,1,1]} + 2\chi_{[2,2,2,2,1]} + \chi_{[2,2,2,1,1,1]} \end{aligned} \quad (9)$$

Here χ_R is the character of the irreducible representation, which is the Schur function in the case of $SU(N)$, while R 's in this case are labelled by the Young diagrams. Hence, from now on, we identify the representations with the Young diagrams. In fact, we need the representation theory of $SU_q(N)$, but it is basically the same at q non-equal to a root of unity.

The coefficients C_{R_1, R_2}^Q count how many times the irreducible representation Q appears in the decomposition, therefore they determine the size of the corresponding Racah matrix. Decomposition (9) shows us that there are two matrices of the size 9×9 , two matrices of the size 8×8 , four matrices of the size 6×6 , four matrices of the size 4×4 , six matrices of the size 2×2 and four trivial "matrices" of the size 1×1 .

We calculate the Racah matrix by definition, i.e. as a transformation matrix from the orthonormal basis (I) to the orthonormal basis (II). To this end, we construct the highest weight vectors in the basis (I) for each representation Q from (9) and same in the basis (II). To proceed, we need manifestly describe the action of lowering and raising operators T_k^\pm on representations of $SU_q(N)$.

To this end, we use the Schur-Weyl duality and, first, realize the representation of $U_q(sl_N)$ in the space of tensors. For each representation labeled by Young diagram $Y = \{Y_1 \geq Y_2 \geq \dots \geq Y_l > 0\}$, we associate the following tensor with all possible permutations of indices:

$$V_{i_1, \dots, i_{Y_1}, j_1, \dots, j_{Y_2}, k_1, \dots, k_{Y_3}, \dots}, \text{ where} \quad (10)$$

$$i_1 = \dots = i_{Y_1} = 0, \quad j_1 = \dots = j_{Y_2} = 0, \quad k_1 = \dots = k_{Y_3} = 2, \dots, \quad (11)$$

in other words, the number of zeros is equal to Y_1 , the number of units is equal to Y_2 , the number of deuces is equal to Y_3 and so on. Thus, every vector of the representation Y can be written as a linear combination of elements (10).

Second, let us define the action of lowering and raising operators T_k^\pm . It is clear that for 1-tensors they act as follows:

$$\begin{aligned} T_k^+ : V_i &\longrightarrow V_{i+1} \\ T_k^- : V_i &\longrightarrow V_{i-1}. \end{aligned} \quad (12)$$

To extend this action to higher rank tensors, one needs a uniquely defined comultiplication Δ on $SU_q(N)$:

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes q^{H_\alpha}, \quad (13)$$

$$\Delta(F_i) = F_i \otimes 1 + q^{-H_\alpha} \otimes F_i, \quad (14)$$

$$\Delta(q^{H_\alpha}) = q^{H_\alpha} \otimes q^{H_\alpha}, \quad (15)$$

where E_i, F_i, q^{H_α} are generators of $SU_q(N)$. Then, for 2-tensors one gets

$$\begin{aligned} T_k^+ : V_{i,j} &\longrightarrow V_{i,j+1} + q^{H_k} V_{i+1,j} \\ T_k^- : V_{i,j} &\longrightarrow q^{-H_k} V_{i,j-1} + V_{i-1,j}. \end{aligned} \quad (16)$$

Since Δ is co-associative, it is easy to extend T_k^\pm actions to any rank tensors.

Now we are ready to construct highest weight vectors. Let us start with $R = [2, 1]$ emerging in the decomposition $[2, 1] \in [1]^{\otimes 3}$:

$$([1] \otimes [1]) \otimes [1] = ([2] + [1, 1]) \otimes [1] = ([3] + [2, 1]) + ([2, 1] + [1, 1, 1]). \quad (17)$$

These decompositions are also can be found by the Littlewood-Richardson rule, and the order of brackets $([1] \otimes [1]) \otimes [1]$ or $[1] \otimes ([1] \otimes [1])$ is not important here, just we should fix some particular order for this case once and forever. It is also not important which particular representation $[2, 1]$ in (17) we choose: the one which comes from $[2] \otimes [1]$ or from $[1, 1] \otimes [1]$. We prefer $[2, 1] \in [1, 1] \otimes [1]$, because our calculations are slightly simpler in this case. Now let us construct the highest weight vector for this representation.

It is clear that the highest weight vector of the fundamental representation $R = [1]$ is just V_0 , because its indices correspond to $R = [1]$ (i.e. $\#0 = 1$, $\#i = 0 \forall i > 1$) and any T_k^- cancels it. In the same way, it is clear that $V_{0,0}$ is the highest weight vector for the symmetric representation $R = [2]$. Now with the help of this vector let us construct the highest weight vector of $R = [1, 1]$. First, one takes tensor product of the two fundamental representations and uses the raising operators to create the corresponding representation:

$$\begin{aligned} T_1^+ (V_0) \otimes V_0 &= V_1 \otimes V_0 = V_{1,0}, \\ V_0 \otimes T_1^+ (V_0) &= V_0 \otimes V_1 = V_{0,1}. \end{aligned} \quad (18)$$

It is possible to create only two different vectors corresponding to $R = [1, 1]$. They form a 2-dimensional vector space with coefficients in $\mathbb{Z}[q, q^{-1}]$. The highest weight vector lies in this space. To determine it, one requires that all lowering operators T_k^- cancel on arbitrary vector from this space:

$$T_k^- (c_1 V_{1,0} + c_2 V_{0,1}) = 0 \Rightarrow c_1 = c, c_2 = -q \cdot c, \quad (19)$$

where c is an arbitrary constant. By definition, the Racah matrix is a transformation matrix from one orthonormal basis to another one, hence, all the highest weight vectors have to be unit vectors. This constraint determines c uniquely

$$c = \frac{1}{\sqrt{1+q^2}}. \quad (20)$$

Thus, we are done already, because $[1, 1] \otimes [1]$ corresponds to $[2, 1]$ itself and is canceled by all T_k^- :

$$\frac{1}{\sqrt{1+q^2}} \cdot (V_{1,0} - qV_{0,1}) \otimes V_0 = \frac{1}{\sqrt{1+q^2}} \cdot (V_{1,0,0} - qV_{0,1,0}). \quad (21)$$

The described procedure allows us to construct the highest weight vector of any representation R step by step. In particular, we found this way the highest weight vectors of all representations Q in both bases (7). These vectors are available at [3], here let us give one simple example.

Example of $Q = [6, 2, 1]$. This representation comes from $[4, 2] \otimes [2, 1]$ and $[4, 1, 1] \otimes [2, 1]$. The corresponding highest weight vectors in the (I) basis are

$$\begin{aligned} & [4, 2] : \\ & \left(-qV_{1,0,0,2,0,0,0,1,0} - q^2V_{2,0,0,1,0,0,0,1,0} - qV_{0,1,0,2,0,0,1,0,0} - q^2V_{0,2,0,1,0,0,1,0,0} - qV_{1,0,0,0,2,0,1,0,0} - q^2V_{2,0,0,0,1,0,1,0,0} \right. \\ & - q^4V_{0,2,0,0,1,0,0,1,0} + q^2V_{0,1,0,2,0,0,1,0} + q^3V_{0,2,0,1,0,0,0,1,0} + q^2V_{1,0,0,0,2,0,0,1,0} + q^3V_{2,0,0,0,1,0,0,1,0} + q^4V_{0,1,0,0,1,0,0,2,0} \\ & - q^3V_{0,1,0,1,0,0,0,2,0} - q^3V_{1,0,0,0,1,0,0,2,0} + q^2V_{1,0,0,1,0,0,0,2,0} - q^3V_{0,1,0,0,1,0,2,0,0} + q^2V_{0,1,0,1,0,0,2,0,0} + q^2V_{1,0,0,0,1,0,2,0,0} \\ & - qV_{1,0,0,1,0,0,2,0,0} + q^6V_{0,1,0,0,1,0,0,2,0} - q^5V_{0,1,0,1,0,0,0,2,0} - q^5V_{1,0,0,0,1,0,0,2,0} + q^4V_{1,0,0,1,0,0,0,2,0} - q^5V_{0,1,0,0,1,0,2,0,0} \\ & \left. + q^4V_{0,1,0,1,0,0,2,0,0} + q^4V_{1,0,0,0,1,0,2,0,0} - q^3V_{1,0,0,1,0,0,2,0,0} + q^2V_{0,1,0,0,2,0,1,0,0} + q^3V_{0,2,0,0,1,0,1,0,0} + V_{1,0,0,2,0,0,1,0,0} \right. \\ & \left. + qV_{2,0,0,1,0,0,1,0,0} - q^3V_{0,1,0,0,2,0,0,1,0} \right) \cdot \frac{1}{\sqrt{q^4+q^2+1}(q^2+1)^2}; \\ & [4, 1, 1] : \\ & \left(q^4V_{0,1,0,0,2,0,0,1,0} - q^3V_{0,1,0,2,0,0,0,1,0} - q^3V_{0,2,0,0,1,0,0,1,0} + q^2V_{0,2,0,1,0,0,0,1,0} - q^3V_{1,0,0,0,2,0,0,1,0} + q^2V_{1,0,0,2,0,0,0,1,0} \right. \\ & + q^2V_{2,0,0,0,1,0,0,1,0} - qV_{2,0,0,1,0,0,0,1,0} - q^3V_{0,1,0,0,2,0,1,0,0} + q^2V_{0,1,0,2,0,0,1,0,0} + q^2V_{0,2,0,0,1,0,1,0,0} - qV_{0,2,0,1,0,0,1,0,0} \\ & \left. + q^2V_{1,0,0,0,2,0,1,0,0} - qV_{1,0,0,2,0,0,1,0,0} - qV_{2,0,0,0,1,0,1,0,0} + V_{2,0,0,1,0,0,1,0,0} \right) \cdot \frac{1}{(q^2+1)^2} \end{aligned}$$

and in the (II) basis

$$\begin{aligned} & [4, 2] : \\ & \left(-q^3V_{0,2,0,0,1,0,0,1,0} + q^2V_{0,2,0,0,1,0,1,0,0} + q^2V_{0,2,0,1,0,0,0,1,0} + q^2V_{2,0,0,0,1,0,0,1,0} - q^5V_{0,2,0,0,1,0,0,1,0} + q^4V_{2,0,0,0,1,0,0,1,0} \right. \\ & + q^4V_{0,2,0,0,1,0,1,0,0} - q^3V_{2,0,0,0,1,0,1,0,0} + q^4V_{0,2,0,1,0,0,0,1,0} - q^3V_{2,0,0,1,0,0,0,1,0} - q^3V_{0,2,0,1,0,0,1,0,0} + q^2V_{2,0,0,1,0,0,1,0,0} \\ & + q^5V_{0,1,0,0,1,0,0,2,0} - q^4V_{1,0,0,0,1,0,0,2,0} - q^4V_{0,1,0,0,1,0,2,0,0} + q^3V_{1,0,0,0,1,0,2,0,0} + q^6V_{0,1,0,0,2,0,0,1,0} - q^5V_{1,0,0,0,2,0,0,1,0} \\ & - q^5V_{0,1,0,0,2,0,1,0,0} + q^4V_{1,0,0,0,2,0,1,0,0} - q^4V_{0,1,0,1,0,0,0,2,0} + q^3V_{1,0,0,1,0,0,0,2,0} + q^3V_{0,1,0,1,0,0,2,0,0} - q^2V_{1,0,0,1,0,0,2,0,0} \\ & \left. - q^5V_{0,1,0,2,0,0,0,1,0} + q^4V_{1,0,0,2,0,0,0,1,0} + q^4V_{0,1,0,2,0,0,1,0,0} - q^3V_{1,0,0,2,0,0,1,0,0} - qV_{2,0,0,1,0,0,0,1,0} - qV_{0,2,0,1,0,0,1,0,0} \right. \\ & \left. - qV_{2,0,0,0,1,0,1,0,0} + V_{2,0,0,1,0,0,1,0,0} \right) \cdot \frac{1}{\sqrt{q^4+q^2+1}(q^2+1)^2}; \end{aligned}$$

[4, 1, 1] :

$$\begin{aligned} & \left(q^4 V_{0,1,0,0,1,0,0,2,0} - q^3 V_{1,0,0,0,1,0,0,2,0} - q^3 V_{0,1,0,0,1,0,2,0,0} + q^2 V_{1,0,0,0,1,0,2,0,0} - q^3 V_{0,1,0,0,2,0,0,1,0} + q^2 V_{1,0,0,0,2,0,0,1,0} \right. \\ & \quad \left. + q^2 V_{0,1,0,0,2,0,1,0,0} - q V_{1,0,0,0,2,0,1,0,0} - q^3 V_{0,1,0,1,0,0,0,2,0} + q^2 V_{1,0,0,1,0,0,0,2,0} + q^2 V_{0,1,0,1,0,0,2,0,0} - q V_{1,0,0,1,0,0,2,0,0} \right. \\ & \quad \left. + q^2 V_{0,1,0,2,0,0,0,1,0} - q V_{1,0,0,2,0,0,0,1,0} - q V_{0,1,0,2,0,0,1,0,0} + V_{1,0,0,2,0,0,1,0,0} \right) \cdot \frac{1}{(q^2 + 1)^2}. \end{aligned}$$

From their explicit forms we find the following rotation matrix, which is Racah matrix by definition

$$\mathcal{U}_{[6,2,1]} = \begin{pmatrix} \frac{1}{[2]} & -\frac{\sqrt{[3]}}{[2]} \\ -\frac{\sqrt{[3]}}{[2]} & -\frac{1}{[2]} \end{pmatrix} \quad (22)$$

Example of $Q = [3, 2, 1]$. Finally let us discuss the case of multiplicities. Indeed, in the product of two $[2, 1]$ there are two representations $[3, 2, 1]$:

$$\chi_{[2,1]}^2 = \chi_{[4,2]} + \chi_{[4,1,1]} + \chi_{[3,3]} + 2\chi_{[3,2,1]} + \chi_{[3,1,1,1]} + \chi_{[2,2,2]} + \chi_{[2,2,1,1]}. \quad (23)$$

In the calculation of the highest weight vector, one obtains the two-dimensional vector space instead of one-dimensional space like in (19). How to choose the two highest weight vectors corresponding to two different $[3, 2, 1]$ representations? We cannot choose them arbitrarily, because the \mathcal{R} -matrix is diagonal only in a particular basis. It turns out that it is enough to put one more condition to determine two different highest weight vectors of $[3, 2, 1]$ corresponding to the diagonal \mathcal{R} -matrix. It is the following condition: one vector has to belong to the *symmetric* tensor square product $[2, 1]^{\otimes 2}$, while the other one belongs to the *antisymmetric* tensor square product. Indeed, when one decomposes the tensor square of the representation λ into irreducible representations μ_i , every representation μ_i comes either from the symmetric or antisymmetric squares. In terms of the highest weight vectors, it means that the vectors are either symmetric or antisymmetric under the following indices permutation:

$$V_{i_1, \dots, i_{|R|}, i_{|R|+1}, \dots, i_{2|R|}} \longrightarrow V_{i_{|R|+1}, \dots, i_{2|R|}, i_1, \dots, i_{|R|}}, \quad (24)$$

where $|R| = \sum_i R_i$. Thus, all irreducible representations of the tensor square of $[2, 1]$ can be divided into two groups, symmetric and antisymmetric (underlined):

$$l[2, 1] \otimes [2, 1] = [4, 2] + \underline{[4, 1, 1]} + \underline{[3, 3]} + [3, 2, 1] + \underline{[3, 2, 1]} + [3, 1, 1, 1] + [2, 2, 2] + \underline{[2, 2, 1, 1]}. \quad (25)$$

This completes the tool box for our calculations. At the end of this section we list two highest weight vectors of $[3, 2, 1]$, symmetric and antisymmetric:

[3, 2, 1] :

$$\begin{aligned} & q^3 V_{1,0,0,2,0,1} - q^5 V_{0,1,0,2,1,0} + q^4 V_{1,0,0,2,1,0} - q^3 V_{0,1,1,0,2,0} + q^2 V_{1,0,1,0,2,0} + q^2 V_{0,1,1,2,0,0} - q V_{1,0,1,2,0,0} \\ & - q^5 V_{0,1,1,0,2,0} + q^4 V_{1,0,1,0,2,0} + q^4 V_{0,1,1,2,0,0} - q^3 V_{1,0,1,2,0,0} - q^4 V_{0,2,0,0,1,1} + q^3 V_{2,0,0,0,1,1} + q^3 V_{0,2,0,1,0,1} \\ & - q^4 V_{0,1,1,0,2,0} + q^3 V_{1,0,1,0,2,0} + q^3 V_{0,1,1,2,0,0} - q^2 V_{1,0,1,2,0,0} - q^3 V_{0,2,0,0,1,1} + q^2 V_{2,0,0,0,1,1} + q^2 V_{0,2,0,1,0,1} \\ & - q V_{2,0,0,1,0,1} + q^5 V_{0,1,0,0,1,2} - q^4 V_{1,0,0,0,1,2} + q^6 V_{0,1,0,0,2,1} - q^5 V_{1,0,0,0,2,1} - q^4 V_{0,1,0,1,0,2} + q^3 V_{1,0,0,1,0,2} \\ & - q^5 V_{0,1,0,2,0,1} + q^4 V_{1,0,0,2,0,1} + q^2 V_{0,2,1,0,1,0} + q^3 V_{1,2,0,0,1,0} - q V_{2,0,1,0,1,0} - q^2 V_{2,1,0,0,1,0} - q V_{0,2,1,1,0,0} \\ & - q^2 V_{1,2,0,1,0,0} + q V_{2,1,0,1,0,0} - q^5 V_{0,2,0,0,1,1} + q^4 V_{2,0,0,0,1,1} + q^4 V_{0,2,0,1,0,1} - q^3 V_{2,0,0,1,0,1} - q^2 V_{2,0,0,1,0,1} \\ & + q^2 V_{0,1,2,0,1,0} + q^3 V_{0,2,1,0,1,0} - q V_{0,1,2,1,0,0} - q^2 V_{0,2,1,1,0,0} - q V_{1,0,2,0,1,0} - q^2 V_{2,0,1,0,1,0} + q V_{2,0,1,1,0,0} \\ & + q^5 V_{0,1,0,0,2,1} - q^4 V_{1,0,0,0,2,1} + q^6 V_{0,1,0,1,2,0} - q^5 V_{1,0,0,1,2,0} - q^4 V_{0,1,0,2,0,1} + V_{2,0,1,1,0,0} + V_{1,0,2,1,0,0} \end{aligned}$$

[3, 2, 1] :

$$\begin{aligned} & -q^3 V_{1,0,0,2,0,1} + q^5 V_{0,1,0,2,1,0} - q^4 V_{1,0,0,2,1,0} + q^3 V_{0,1,1,0,2,0} - q^2 V_{1,0,1,0,2,0} - q^2 V_{0,1,1,2,0,0} + q V_{1,0,1,2,0,0} \\ & + q^5 V_{0,1,1,0,2,0} - q^4 V_{1,0,1,0,2,0} - q^4 V_{0,1,1,2,0,0} + q^3 V_{1,0,1,2,0,0} + q^4 V_{0,2,0,0,1,1} - q^3 V_{2,0,0,0,1,1} - q^3 V_{0,2,0,1,0,1} \\ & - q^4 V_{0,1,1,0,2,0} + q^3 V_{1,0,1,0,2,0} + q^3 V_{0,1,1,2,0,0} - q^2 V_{1,0,1,2,0,0} - q^3 V_{0,2,0,0,1,1} + q^2 V_{2,0,0,0,1,1} + q^2 V_{0,2,0,1,0,1} \\ & - q V_{2,0,0,1,0,1} + q^5 V_{0,1,0,0,1,2} - q^4 V_{1,0,0,0,1,2} + q^6 V_{0,1,0,0,2,1} - q^5 V_{1,0,0,0,2,1} - q^4 V_{0,1,0,1,0,2} + q^3 V_{1,0,0,1,0,2} \\ & - q^5 V_{0,1,0,2,0,1} + q^4 V_{1,0,0,2,0,1} + q^2 V_{0,2,1,0,1,0} + q^3 V_{1,2,0,0,1,0} - q V_{2,0,1,0,1,0} - q^2 V_{2,1,0,0,1,0} - q V_{0,2,1,1,0,0} \\ & - q^2 V_{1,2,0,1,0,0} + q V_{2,1,0,1,0,0} - q^5 V_{0,2,0,0,1,1} + q^4 V_{2,0,0,0,1,1} + q^4 V_{0,2,0,1,0,1} - q^3 V_{2,0,0,1,0,1} + q^2 V_{2,0,0,1,0,1} \\ & - q^2 V_{0,1,2,0,1,0} - q^3 V_{0,2,1,0,1,0} + q V_{0,1,2,1,0,0} + q^2 V_{0,2,1,1,0,0} + q V_{1,0,2,0,1,0} + q^2 V_{2,0,1,0,1,0} - q V_{2,0,1,1,0,0} \\ & - q^5 V_{0,1,0,0,2,1} + q^4 V_{1,0,0,0,2,1} - q^6 V_{0,1,0,1,2,0} + q^5 V_{1,0,0,1,2,0} + q^4 V_{0,1,0,2,0,1} + V_{2,0,1,1,0,0} - V_{1,0,2,1,0,0} \end{aligned}$$

3 Non-symmetric representation $R = [2, 1]$

In this section we construct the \mathcal{R} -matrices and the mixing matrices for the non-symmetric representation $R = [2, 1]$ for the three strand case in order to evaluate the HOMFLY polynomials in representation $R = [2, 1]$. The polynomials themselves are discussed in further sections. Since our approach is the group theoretical one, the \mathcal{R} -matrices are obtained in the vertical framing.

3.1 Two strands

Since $\chi_{[2,1]}^2 = \chi_{[4,2]} + \chi_{[4,1,1]} + \chi_{[3,3]} + 2\chi_{[3,2,1]} + \chi_{[3,1,1,1]} + \chi_{[2,2,2]} + \chi_{[2,2,1,1]}$,

$$l[2, 1] \otimes [2, 1] = [4, 2] + \underline{[4, 1, 1]} + \underline{[3, 3]} + [3, 2, 1] + \underline{[3, 2, 1]} + [3, 1, 1, 1] + [2, 2, 2] + \underline{[2, 2, 1, 1]} \quad (26)$$

Underlined are representations from antisymmetric square, for which a minus sign should be added to the eigenvalues of the \mathcal{R} -matrices². In particular, the two $[3, 2, 1]$ come from symmetric and antisymmetric squares and thus are well defined basis vectors in the two-dimensional space as it was explained at the end of the previous section. The Casimir (cut-and-join) eigenvalues $\varkappa_Q = \sum_{(i,j) \in Q} (i - j)$ are $\varkappa_{[4,2]} = 5$, $\varkappa_{[4,1,1]} = 3$, $\varkappa_{[3,3]} = 3$, $\varkappa_{[3,2,1]} = 0$, $\varkappa_{[3,1,1,1]} = -3$, $\varkappa_{[2,2,2]} = -3$, $\varkappa_{[2,2,1,1]} = -5$, therefore, the Rosso-Jones formula [34] for the two-strand torus link $T[2, n]$ gives:

$$H_{[2,1] \otimes [2,1]}^{[2,2n]} = q^{10n} \chi_{[4,2]}^* + q^{6n} \chi_{[4,1,1]}^* + q^{6n} \chi_{[3,3]}^* + 2\chi_{[3,2,1]}^* + q^{-6n} \chi_{[3,1,1,1]}^* + q^{-6n} \chi_{[2,2,2]}^* + q^{-10n} \chi_{[2,2,1,1]}^* \quad (28)$$

where the asterisk means that the character is calculated at "the topological locus" $p_k = \frac{A^k - A^{-k}}{q^k - q^{-k}}$.

For two-strand torus knot $T[2, 2n+1]$ one needs slightly more complicated Adams coefficients $\widehat{\text{Ad}}_2 \chi_{[2,1]} = \chi_{[4,2]} - \chi_{[4,1,1]} - \chi_{[3,3]} + \chi_{[3,1,1,1]} + \chi_{[2,2,2]} - \chi_{[2,2,1,1]}$ and

$$H_{[2,1]}^{[2,2n+1]} = q^{5(2n+1)} \chi_{[4,2]}^* - q^{3(2n+1)} \chi_{[4,1,1]}^* - q^{3(2n+1)} \chi_{[3,3]}^* + q^{-3(2n+1)} \chi_{[3,1,1,1]}^* + q^{-3(2n+1)} \chi_{[2,2,2]}^* - q^{-5(2n+1)} \chi_{[2,2,1,1]}^* \quad (29)$$

Note that $\chi_{[3,2,1]}$ does not contribute in this case, which means that the two eigenvalues have different signs.

3.2 Three strands

In the three-strand case, one needs

$$\begin{aligned} l[2, 1]^{\otimes 3} &= ([4, 2] + [4, 1, 1] + [3, 3] + 2 \cdot [3, 2, 1] + [3, 1, 1, 1] + [2, 2, 2] + [2, 2, 1, 1]) \otimes [2, 1] = \\ &= ([6, 3] + [6, 2, 1] + [5, 4] + 2 \cdot [5, 3, 1] + [5, 2, 2] + [5, 2, 1, 1] + [4, 4, 1] + [4, 3, 2] + [4, 3, 1, 1] + [4, 2, 2, 1]) + \\ &+ ([6, 2, 1] + [6, 1, 1, 1] + [5, 3, 1] + [5, 2, 2] + 2 \cdot [5, 2, 1, 1] + [5, 1, 1, 1, 1] + [4, 3, 2] + [4, 3, 1, 1] + [4, 2, 2, 1] + [4, 2, 1, 1, 1]) + \\ &+ ([5, 4] + [5, 3, 1] + [4, 4, 1] + [4, 3, 2] + [4, 3, 1, 1] + [3, 3, 2, 1]) + \\ &+ 2 \cdot ([5, 3, 1] + [5, 2, 2] + [5, 2, 1, 1] + [4, 4, 1] + 2 \cdot [4, 3, 2] + 2 \cdot [4, 3, 1, 1] + 2 \cdot [4, 2, 2, 1] + \\ &+ [4, 2, 1, 1, 1] + [3, 3, 3] + 2 \cdot [3, 3, 2, 1] + [3, 3, 1, 1, 1] + [3, 2, 2, 2] + [3, 2, 2, 1, 1]) + \\ &+ ([5, 2, 1, 1] + [5, 1, 1, 1, 1] + [4, 3, 1, 1] + [4, 2, 2, 1] + 2 \cdot [4, 2, 1, 1, 1] + [4, 1, 1, 1, 1, 1] + [3, 3, 2, 1] + [3, 3, 1, 1, 1] + [3, 2, 2, 1, 1] + [3, 2, 1, 1, 1, 1]) + \\ &+ ([4, 3, 2] + [4, 2, 2, 1] + [3, 3, 2, 1] + [3, 2, 2, 2] + [3, 2, 2, 1, 1] + [2, 2, 2, 2, 1]) + \\ &+ ([4, 3, 1, 1] + [4, 2, 2, 1] + [4, 2, 1, 1, 1] + [3, 3, 2, 1] + [3, 3, 1, 1, 1] + [3, 2, 2, 2] + 2 \cdot [3, 2, 2, 1, 1] + [3, 2, 1, 1, 1, 1] + [2, 2, 2, 2, 1] + [2, 2, 2, 1, 1, 1]) = \end{aligned}$$

²One can determine if the representation belongs to symmetric or antisymmetric square looking at the symmetry of its highest weight vector as it was done in the previous section. However, technically the simplest way to find a decomposition of square of the representation R into symmetric and antisymmetric parts is to use the plethysm: one has to expand

$$\frac{1}{2} \chi_R(p)^2 \pm \frac{1}{2} \chi_R(p^{(2)}) = \sum_{Q_{\pm}} \chi_{Q_{\pm}}(p) \quad (27)$$

in order to find Q_{\pm} 's that emerge in the symmetric (+) and antisymmetric (-) squares. Here $|Q_{\pm}| = 2|R|$, $\chi_R(p)$ is the Schur function of variables $p_k = \text{Tr } g^k$, and the Adams 2-transformation is given by $p_k \rightarrow p_{2k} \equiv p_k^{(2)}$.

$$\begin{aligned}
&= \underbrace{[6, 3]}_{[4, 2]} + \underbrace{2 \cdot [6, 2, 1]}_{[4, 2] + [4, 1, 1]} + \underbrace{[6, 1, 1, 1]}_{[4, 1, 1]} + \underbrace{2 \cdot [5, 4]}_{[4, 2] + [3, 3]} + \underbrace{6 \cdot [5, 3, 1]}_{2 \cdot [4, 2] + [4, 1, 1] + [3, 3] + 2 \cdot [3, 2, 1]} + \underbrace{4 \cdot [5, 2, 2]}_{[4, 2] + [4, 1, 1] + 2 \cdot [3, 2, 1]} + \underbrace{6 \cdot [5, 2, 1, 1]}_{[4, 2] + 2 \cdot [4, 1, 1] + 2 \cdot [3, 2, 1] + [3, 1, 1, 1]} + \\
&\quad + \underbrace{2 \cdot [5, 1, 1, 1, 1]}_{[4, 1, 1] + [3, 1, 1, 1]} + \underbrace{4 \cdot [4, 4, 1]}_{[4, 2] + [3, 3] + 2 \cdot [3, 2, 1]} + \underbrace{8 \cdot [4, 3, 2]}_{[4, 2] + [4, 1, 1] + [3, 3] + 4 \cdot [3, 2, 1] + [2, 2, 2]} + \underbrace{9 \cdot [4, 3, 1, 1]}_{[4, 2] + [4, 1, 1] + [3, 3] + 4 \cdot [3, 2, 1] + [3, 1, 1, 1] + [2, 2, 1, 1]} + \\
&\quad + \underbrace{9 \cdot [4, 2, 2, 1]}_{[4, 2] + [4, 1, 1] + 4 \cdot [3, 2, 1] + [3, 1, 1, 1] + [2, 2, 2] + [2, 2, 1, 1]} + \underbrace{6 \cdot [4, 2, 1, 1, 1]}_{[4, 1, 1] + 2 \cdot [3, 2, 1] + 2 \cdot [3, 1, 1, 1] + [2, 2, 1, 1]} + \underbrace{[411111]}_{[3111]} + \\
&\quad + \underbrace{2 \cdot [3, 3, 3]}_{2 \cdot [3, 2, 1]} + \underbrace{8 \cdot [3, 3, 2, 1]}_{[3, 3] + 4 \cdot [3, 2, 1] + [3, 1, 1, 1] + [2, 2, 2] + [2, 2, 1, 1]} + \underbrace{4 \cdot [3, 3, 1, 1, 1]}_{2 \cdot [3, 2, 1] + [3, 1, 1, 1] + [2, 2, 1, 1]} + \underbrace{4 \cdot [3, 2, 2, 2]}_{2 \cdot [3, 2, 1] + [2, 2, 2] + [2, 2, 1, 1]} + \\
&\quad + \underbrace{6 \cdot [3, 2, 2, 1, 1]}_{2 \cdot [3, 2, 1] + [3, 1, 1, 1] + [2, 2, 2] + 2 \cdot [2, 2, 1, 1]} + \underbrace{2 \cdot [3, 2, 1, 1, 1, 1]}_{[3, 1, 1, 1] + [2, 2, 1, 1]} + \underbrace{2 \cdot [2, 2, 2, 2, 1]}_{[2, 2] + [2, 2, 1, 1]} + \underbrace{[2, 2, 2, 1, 1, 1]}_{[2, 2, 1, 1]}
\end{aligned}$$

For the three-strand torus links one has ($n = 3k \pm 1$):

$$\begin{aligned}
H_{[2,1]}^{[3,n]} &= q^{-10n} \chi_{[6,3]} - q^{-8n} \chi_{[6,2,1]}^* + q^{-6n} \chi_{[6,1,1,1]}^* - q^{-8n} \chi_{[5,4]}^* + q^{-4n} \chi_{[5,2,2]}^* - \chi_{[5,1,1,1,1]}^* + q^{-4n} \chi_{[4,4,1]}^* - q^{-2n} \chi_{[4,3,2]}^* + \\
&+ q^{6n} \chi_{[4,1,1,1,1,1]}^* + 2 \chi_{[3,3,3]}^* - q^{2n} \chi_{[3,3,2,1]}^* + q^{4n} \chi_{[3,3,1,1,1]}^* + q^{4n} \chi_{[3,2,2,2]}^* - q^{8n} \chi_{[3,2,1,1,1,1]}^* - q^{8n} \chi_{[2,2,2,2,1]}^* + q^{10n} \chi_{[2,2,2,1,1,1]}^*
\end{aligned}$$

3.3 Mixing matrices

The 1×1 \mathcal{R} -matrices are

$$\mathcal{R}_{[6,3]} = q^{-\varkappa_{[4,2]}} = q^{-5} \quad (30)$$

and

$$\mathcal{R}_{[6,1,1]} = -q^{\varkappa_{[4,1,1]}} = -q^{-3} \quad (31)$$

As for the higher size mixing matrices, one can calculate them using the results of the previous section. However, for the sizes of matrices up to five there is the eigenvalue conjecture [24] that allows one to obtain the mixing matrices immediately. That is, according to the eigenvalue hypothesis of [24] (see also [43]) the 2×2 and 4×4 mixing matrices are expressed through the normalized eigenvalues of \mathcal{R} -matrix $\tilde{\xi}_Q$:

$$\tilde{\xi}_i = \frac{\xi_i}{\left(\pm \prod_{j=1}^p \xi_j\right)^{1/p}} \quad (32)$$

where the product runs over all $Q_j \in R \otimes R$, $j = 1 \dots p$. For 2×2 mixing matrices, [24] conjectures:

$$U_{ii} = \frac{\sqrt{-1}}{\prod_{k \neq i} \tilde{\xi}_{ik}}, \quad U_{ij} = \sqrt{\frac{(\tilde{\xi}_i^2 - 1 + \tilde{\xi}_i^{-2})}{\tilde{\xi}_{ij}^2 \tilde{\xi}_{ik} \tilde{\xi}_{jk}}} = \sqrt{\frac{(\tilde{\xi}_j^2 - 1 + \tilde{\xi}_j^{-2})}{\tilde{\xi}_{ij}^2 \tilde{\xi}_{ik} \tilde{\xi}_{jk}}} \quad (33)$$

while for 4×4 mixing matrices, [24] conjectures:

$$U_{ii} = \frac{\tilde{\xi}_i^2 \sum_{(k < l) \neq i} \tilde{\xi}_k \tilde{\xi}_l - \tilde{\xi}_i \sum_{k \neq i} \tilde{\xi}_k}{\prod_{k \neq i} \tilde{\xi}_{ik}}, \quad U_{ij} = \sqrt{\frac{(\tilde{\xi}_i^2 - 1)(\tilde{\xi}_j^2 - 1)}{\tilde{\xi}_{ij}^2}} \prod_{k \neq i, j} \left(\frac{\tilde{\xi}_i \tilde{\xi}_k - 1 + (\tilde{\xi}_i \tilde{\xi}_k)^{-1}}{\tilde{\xi}_{ik} \tilde{\xi}_{jk}} \right) \quad (34)$$

(making use of $\prod_{k=1}^4 \tilde{\xi}_k$ one can check that the last product is symmetric under the permutation $i \leftrightarrow j$).

Using these formulas, one easily gets

$$\mathcal{R}_{[6,2,1]} = \begin{pmatrix} q^{-\varkappa_{[4,2]}} & \\ & -q^{-\varkappa_{[4,1,1]}} \end{pmatrix} = \begin{pmatrix} \frac{1}{q^5} & \\ & -\frac{1}{q^3} \end{pmatrix} \quad \mathcal{U}_{[6,2,1]} = \begin{pmatrix} \frac{1}{[2]} & -\frac{\sqrt{[3]}}{[2]} \\ -\frac{\sqrt{[3]}}{[2]} & -\frac{1}{[2]} \end{pmatrix} \quad (35)$$

$$\mathcal{R}_{[5,4]} = \begin{pmatrix} q^{-\varkappa_{[4,2]}} & \\ & -q^{-\varkappa_{[3,3]}} \end{pmatrix} = \begin{pmatrix} \frac{1}{q^5} & \\ & -\frac{1}{q^3} \end{pmatrix} \quad \mathcal{U}_{[5,4]} = \begin{pmatrix} \frac{1}{[2]} & -\frac{\sqrt{[3]}}{[2]} \\ -\frac{\sqrt{[3]}}{[2]} & -\frac{1}{[2]} \end{pmatrix} \quad (36)$$

$$\mathcal{R}_{[5,1,1,1,1]} = \begin{pmatrix} -q^{-\mathcal{K}_{[4,1,1]}} & & & \\ & q^{-\mathcal{K}_{[3,1,1,1]}} & & \\ & & & \\ & & & \end{pmatrix} = \begin{pmatrix} \frac{1}{q^5} & & & \\ & -\frac{1}{q^3} & & \\ & & & \\ & & & \end{pmatrix} \quad \mathcal{U}_{[5,1,1,1,1]} = \begin{pmatrix} \frac{1}{q^3+q^{-3}} & \frac{\sqrt{q^6+1+q^{-6}}}{q^3+q^{-3}} \\ \frac{\sqrt{q^6+1+q^{-6}}}{q^3+q^{-3}} & -\frac{1}{q^3+q^{-3}} \\ & & & \\ & & & \end{pmatrix} \quad (37)$$

$$\mathcal{R}_{[3,3,3]} = \begin{pmatrix} q^{-\mathcal{K}_{[3,2,1]}} & & \\ & -q^{-\mathcal{K}_{[3,2,1]}} & \\ & & \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix} \quad \mathcal{U}_{[3,3,3]} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (38)$$

$$\mathcal{R}_{[5,2,2]} = \begin{pmatrix} q^{-\mathcal{K}_{[4,2]}} & & & \\ & -q^{-\mathcal{K}_{[4,1,1]}} & & \\ & & -q^{-\mathcal{K}_{[3,2,1]}} & \\ & & & q^{-\mathcal{K}_{[3,2,1]}} \end{pmatrix} = \begin{pmatrix} \frac{1}{q^5} & & & \\ & -\frac{1}{q^3} & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \quad (39)$$

$$U_{[5,2,2]} = \begin{pmatrix} -\frac{1}{[2][5]} & \frac{1}{[2]}\sqrt{\frac{[3]}{[5]}} & -\beta & \alpha \\ \frac{1}{[2]}\sqrt{\frac{[3]}{[5]}} & \frac{1}{[2]} & \delta & \gamma \\ \beta & -\delta & \frac{[1/2]\cdot\{q\}^2}{2(q^{5/2}+q^{-5/2})} & \frac{[2]}{2\sqrt{[5]}} \\ \alpha & \gamma & -\frac{[2]}{2\sqrt{[5]}} & -\frac{q^{1/2}+q^{-1/2}}{2\cdot[5/2]} \end{pmatrix} \quad (40)$$

with

$$\alpha = \sqrt{\frac{q^4+1+q^{-4}}{2}} \frac{\sqrt{[3](q+1+q^{-1})}}{[5]} \sqrt{\frac{q^{5/2}+q^{-5/2}}{q^{3/2}+q^{-3/2}}} \quad \beta = \sqrt{\frac{q^4+1+q^{-4}}{2}} \sqrt{\frac{q-1+q^{-1}}{[5]}} \sqrt{\frac{q^{3/2}+q^{-3/2}}{q^{5/2}+q^{-5/2}}} \\ \gamma = \sqrt{\frac{q^4+1+q^{-4}}{2}} \sqrt{\frac{q-1+q^{-1}}{[3][5]}} \sqrt{\frac{q^{5/2}+q^{-5/2}}{q^{3/2}+q^{-3/2}}} \quad \delta = \sqrt{\frac{q^4+1+q^{-4}}{2}} \frac{\sqrt{q+1+q^{-1}}}{[3]} \sqrt{\frac{q^{3/2}+q^{-3/2}}{q^{5/2}+q^{-5/2}}}$$

so that

$$\gamma^2 + \delta^2 = \frac{[6]}{[2][5]}, \quad \gamma^2 - \delta^2 = \frac{[6]}{[3][5]}, \quad 2\gamma\delta = \frac{[6]}{[2][3]\sqrt{[5]}} \quad (41)$$

$$\mathcal{R}_{[4,4,1]} = \begin{pmatrix} q^{-\mathcal{K}_{[4,2]}} & & & \\ & -q^{-\mathcal{K}_{[3,3]}} & & \\ & & -q^{-\mathcal{K}_{[3,2,1]}} & \\ & & & q^{-\mathcal{K}_{[3,2,1]}} \end{pmatrix} = \begin{pmatrix} \frac{1}{q^5} & & & \\ & -\frac{1}{q^3} & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \quad (42)$$

$$\mathcal{U}_{[4,4,1]} = \mathcal{U}_{[5,2,2]} \quad (43)$$

For 6×6 , 8×8 and 9×9 matrices the eigenvalues of \mathcal{R} do not define the mixing matrix uniquely, thus, the mixing matrices need to be calculated by the direct method of [2]. The result is listed in Appendix A.

3.4 Euler angles

The mixing matrices usually have a simple parameterization in terms of elementary buildings block associated with simple choices of Euler angles. For instance, using formulas (41), one has perform a rotation at angle θ with $c = \cos \theta = \frac{\delta}{\sqrt{\gamma^2 + \delta^2}}$ and $s = \sin \theta = \frac{\gamma}{\sqrt{\gamma^2 + \delta^2}}$ in the $[3, 2, 1]$ sector that converts the off-diagonal 2×2 block of $U_{[5,2,2]}$ into:

$$\begin{aligned} \alpha s - \beta c &= \frac{\alpha\gamma - \beta\delta}{\sqrt{\gamma^2 + \delta^2}} = \frac{1}{[5]} \sqrt{\frac{[6]}{[2][3]}} & \alpha c + \beta s &= \frac{\alpha\delta + \beta\gamma}{\sqrt{\gamma^2 + \delta^2}} = \sqrt{\frac{[6][2]}{[3][5]}} \\ \gamma s + \delta c &= \sqrt{\frac{[6]}{[2][5]}} & \gamma c - \delta s &= 0 \\ \alpha s + \beta c &= \frac{\alpha\gamma + \beta\delta}{\sqrt{\gamma^2 + \delta^2}} = \frac{[4]}{[5]} \sqrt{\frac{[6]}{[2][3]}} & \alpha c - \beta s &= \frac{\alpha\delta - \beta\gamma}{\sqrt{\gamma^2 + \delta^2}} = \sqrt{\frac{[6]}{[2][3][5]}} \\ \gamma s - \delta c &= \frac{\gamma^2 - \delta^2}{\sqrt{\gamma^2 + \delta^2}} = \frac{1}{[3]} \sqrt{\frac{[6][2]}{[5]}} & \gamma c + \delta s &= \frac{2\gamma\delta}{\sqrt{\gamma^2 + \delta^2}} = \frac{1}{[3]} \sqrt{\frac{[6]}{[2]}} \end{aligned} \quad (44)$$

Thus, one gets the matrix

$$\tilde{U}_{[5,2,2]} = \begin{pmatrix} -\frac{1}{[2][5]} & \frac{1}{[2]}\sqrt{\frac{[3]}{[5]}} & \frac{1}{[5]}\sqrt{\frac{[6]}{[3][2]}} & \sqrt{\frac{[6][2]}{[5][3]}} \\ \frac{1}{[2]}\sqrt{\frac{[3]}{[5]}} & \frac{1}{[2]} & -\sqrt{\frac{[6]}{[5][2]}} & 0 \\ \frac{1}{[5]}\sqrt{\frac{[6]}{[3][2]}} & -\sqrt{\frac{[6]}{[5][2]}} & \frac{[6]}{[5][3]} & \sqrt{\frac{1}{[5]}} \\ \sqrt{\frac{[6][2]}{[5][3]}} & 0 & \sqrt{\frac{1}{[5]}} & 0 \end{pmatrix} \quad (45)$$

related with $U_{[5,2,2]}$ by the simple rotation:

$$\tilde{U}_{[5,2,2]} = V^{-1}U_{[5,2,2]}V$$

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin(x) & \cos(x) \\ 0 & 0 & -\cos(x) & \sin(x) \end{pmatrix} \quad (46)$$

with $\sin(x) = \sqrt{\frac{q^4+q^2+1+q^3+q}{2(q^4+q^2+1)}}$, $\cos(x) = \sqrt{\frac{q^4+q^2+1-q^3-q}{2(q^4+q^2+1)}}$.

This new mixing matrix can be further decomposed into the product of elementary 2×2 matrices of rotation:

$$\tilde{U}_{[5,2,2]} = r_1 r_2 r_3 r_4 r_5 r_6, \quad (47)$$

where

$$r_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(x_1) & -\sin(x_1) \\ 0 & 0 & \sin(x_1) & \cos(x_1) \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(x_2) & 0 & \sin(x_2) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(x_2) & 0 & \cos(x_2) \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(x_3) & -\sin(x_3) & 0 \\ 0 & \sin(x_3) & \cos(x_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (48)$$

$$r_4 = \begin{pmatrix} \cos(x_4) & 0 & 0 & -\sin(x_4) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin(x_4) & 0 & 0 & \cos(x_4) \end{pmatrix}, \quad r_5 = \begin{pmatrix} \cos(x_5) & 0 & \sin(x_5) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(x_5) & 0 & \cos(x_5) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad r_6 = \begin{pmatrix} \cos(x_6) & -\sin(x_6) & 0 & 0 \\ \sin(x_6) & \cos(x_6) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (49)$$

and the corresponding angles are

$$\begin{aligned} \cos(x_1) &= \sqrt{\frac{1}{[5]}}, & \cos(x_2) &= \frac{[4]}{[2]\sqrt{[5]}}, & \cos(x_3) &= \frac{1}{[4]}, & \cos(x_4) &= 1, & \cos(x_5) &= 0, & \cos(x_6) &= \frac{[2]}{[4]} \\ \sin(x_1) &= \sqrt{\frac{[2][6]}{[3][5]}}, & \sin(x_2) &= \sqrt{\frac{[6]}{[2][3][5]}}, & \sin(x_3) &= \frac{\sqrt{[3][5]}}{[4]}, & \sin(x_4) &= 0, & \sin(x_5) &= 1, & \sin(x_6) &= \frac{\sqrt{[2][6]}}{[4]} \end{aligned} \quad (50)$$

4 Attaching fingers

The ingredients that we constructed in the previous sections, that is, the \mathcal{R} -matrices and the 3-strand mixing matrices allow one to immediately evaluate the HOMFLY polynomials of knots/links presented by 3-strand braids via the product of matrices:

$$H_{[2,1]}^\mathcal{K} = \sum_Q \chi_Q^* \cdot \text{Tr} \left(\mathcal{R}_Q^{a_1} \left(\mathcal{U}_Q \mathcal{R}_Q \tilde{\mathcal{U}}_Q \right)^{b_1} \mathcal{R}_Q^{a_2} \left(\mathcal{U}_Q \mathcal{R}_Q \tilde{\mathcal{U}}_Q \right)^{b_2} \dots \right) \quad (51)$$

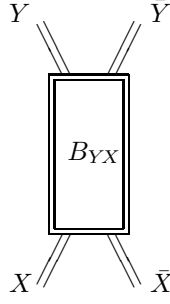


Figure 1: An elementary building block (propagator) for fingers.

where Tr is just the matrix trace, tilde means matrix transposition and the sum goes over all Q lying in $[2, 1]^{\otimes 3}$.

However, the variety of knots/links that can be described by a closed 3-strand braid is not that large. It can be considerably enlarged by allowance to insert instead of \mathcal{R} -matrices arbitrary "double-fat fingers" F_X , $X \in R \otimes R$. The finger is the building block $B_{X,Y}$, Figure 1 with the top external double-lines closed with each other, i.e. $F_X = B_{0,X}$, the details can be found in [28], where the fingers under consideration are called parallel. It is nothing but the plat representation of the two-bridge knot with the two ending arcs cut³. The propagator $B_{X,Y}$ can be also defined as a 4-strand braid with two parallel and two antiparallel strands.

In the case of two strand braids, one just inserts F_X instead of \mathcal{R}_Q : the \mathcal{R} -matrix is diagonal and each diagonal entry R_X corresponds to a representation $X \in R \otimes R$. We substitute it with F_X and call the corresponding diagonal matrix \mathcal{F}_Q . This was done for representation $[2, 1]$ in [28]. In the 3-strand case, one inserts \mathcal{F}_Q and $\mathcal{U}_Q \mathcal{F}_Q \tilde{\mathcal{U}}_Q$. This is what was done in [1] for simpler representations and what we do here for representation $[2, 1]$.

The fingers can be chosen arbitrary. As an example, we considered in [1] a 7-parametric family that involved seven possible fingers, four of them being just pretzel fingers:

$$\begin{aligned} F_X^1(n) &= \frac{(\bar{S}\bar{T}^n S)_{0,X}}{S_{0,X}} \\ F_X^2(n) &= \frac{(ST^2 S^\dagger \bar{T}^{n_{1,6}} S)_{0,X}}{S_{0,X}} \\ F_X^{3,\pm}(n) &= \frac{(\bar{S}\bar{T}^{\pm 2} \bar{S}\bar{T}^n S)_{0,X}}{S_{0,X}} \end{aligned} \quad (52)$$

The matrices T, \bar{T} are in fact the diagonal \mathcal{R} -matrices in the singlet $\in [2, 1]^{\otimes 2} \otimes \overline{[2, 1]}^{\otimes 2}$ (T corresponds to first two parallel strands and \bar{T} to first two antiparallel) and S, \bar{S} the corresponding mixing matrices. They can be found in Appendix B, see [28] for details.

Let us stress here that though this case of two parallel and two antiparallel strands may look more complicated, since the mixing matrices manifestly depend on N (or $A = q^N$), in the way we use them, there is no sum over many $Q \in [2, 1] \otimes [2, 1] \otimes [2, 1]$ or $Q \in [2, 1] \otimes [2, 1] \otimes [2, 1]$, since only $Q = [2, 1]$ contributes. Hence, one suffices to calculate only two mixing matrices S and \bar{S} in contrast with 17 mixing matrices of the previous section.

Thus, following the example in [1], instead of (51) we consider a 7-parametric family (see Figure 2)

$$\begin{aligned} &\chi_{[2,1]}^* H_{[2,1]}^{(n_1, \dots, n_7 | \pm)} = \\ &= \sum_{Q \in [2,1]^{\otimes 2}} \chi_Q^* \cdot \text{Tr} \left[\mathcal{F}_Q^1(n_1) \mathcal{F}_Q^2(n_2) \left(\mathcal{U}_Q \mathcal{F}_Q^2(n_3) \tilde{\mathcal{U}}_Q \right) \mathcal{F}_Q^2(n_4) \left(\mathcal{U}_Q \mathcal{F}_Q^2(n_5) \tilde{\mathcal{U}}_Q \right) \mathcal{F}_Q^1(n_6) \left(\mathcal{U}_Q \mathcal{F}_Q^{3,\pm}(n_7) \tilde{\mathcal{U}}_Q \right) \right] \end{aligned} \quad (53)$$

This family covers almost all the knots from the Rolfsen table, [37]. We discussed these HOMFLY polynomials in the next section.

A few comments are in order. First of all, in all manifest further formulas we use the topological framing, which means that one has renormalize all the \mathcal{R} -matrices with the factor A^3 (and with $A^{|R|}$ in the generic

³In fact, the class of graphs under consideration is immediately extended in order to include also those with full propagators $B_{Y,X}$, see [1].

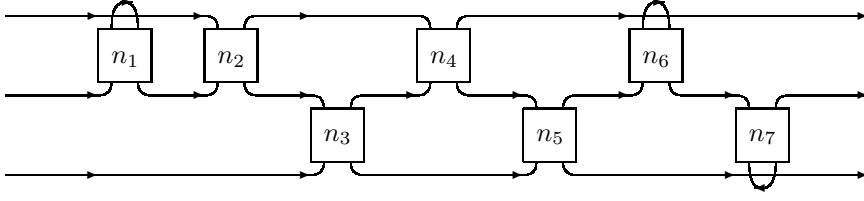


Figure 2: 7-parametric family that covers most of the knots in the Rolfsen table.

representation R , cf. (4)). Second, we normalize our HOMFLY polynomials H_R dividing by the corresponding HOMFLY polynomial of the unknot, which is nothing but the quantum dimensions χ_R^* . Third, in the examples of the knots represented by a 3-braid with fingers which we considered so far, we did not need any additional sign factors even for the knots that unavoidably required introducing non-trivial sign factors in [28]. The reason is that contributions of the finger components that required non-trivial sign factors to be treated correctly in [28] (components non-diagonal in multiplicities) do not contribute in the case of fingers of type (52).

5 Polynomials

As we already mentioned, the 7-parametric family, Figure 2 covers almost the whole Rolfsen table, [37] as was demonstrated in [1, Appendix]. Using (53) and the manifest formulas for the mixing matrices, s.3 and Appendix A, one can calculate the colored HOMFLY polynomials in representation $[2, 1]$ for these knots. The answers will appear in [3]. In fact, most of data concerning the $[2, 1]$ -colored HOMFLY polynomials will be appearing in [3], and some has been already published in [26, 32, 28, 27]. Here for an illustrative purpose we just give an example of the $[2, 1]$ -colored HOMFLY polynomial for the enough complicated knot with 10 crossings, 10_{89} which is described by at least a 5-strand closed braid (without fingers)⁴, and has never been evaluated before:

$$\begin{aligned}
H_{[21]}^{10_{89}} = & \frac{1}{q^{30}} \cdot [(-q^{20} + 8q^{18} - 27q^{16} + 57q^{14} - 101q^{12} + 169q^{10} - 253q^8 + 332q^6 - 399q^4 + 454q^2 - 477 + 454q^{-2} - 399q^{-4} + 332q^{-6} - 253q^{-8} + 169q^{-10} - \\
& -101q^{-12} + 57q^{-14} - 27q^{-16} + 8q^{-18} - q^{-20}) + \frac{1}{A^2}(q^{26} - 7q^{24} + 20q^{22} - 35q^{20} + 55q^{18} - 92q^{16} + 140q^{14} - 185q^{12} + 241q^{10} - 315q^8 + 379q^6 - 422q^4 + \\
& + 462q^2 - 484 + 462q^{-2} - 422q^{-4} + 379q^{-6} - 315q^{-8} + 241q^{-10} - 185q^{-12} + 140q^{-14} - 92q^{-16} + 55q^{-18} - 35q^{-20} + 20q^{-22} - 7q^{-24} + q^{-26}) + \\
& + \frac{1}{A^4}(-q^{30} + 8q^{28} - 32q^{26} + 92q^{24} - 217q^{22} + 451q^{20} - 857q^{18} + 1498q^{16} - 2421q^{14} + 3640q^{12} - 5110q^{10} + 6714q^8 - 8295q^6 + 9645q^4 - 10537q^2 + 10841 - \\
& -10537q^{-2} + 9645q^{-4} - 8295q^{-6} + 6714q^{-8} - 5110q^{-10} + 3640q^{-12} - 2421q^{-14} + 1498q^{-16} - 857q^{-18} + 451q^{-20} - 217q^{-22} + 92q^{-24} - 32q^{-26} + 8q^{-28} - \\
& -q^{-30}) + \frac{1}{A^6}(4q^{30} - 30q^{28} + 111q^{26} - 298q^{24} + 684q^{22} - 1410q^{20} + 2623q^{18} - 4446q^{16} + 6974q^{14} - 10201q^{12} + 13956q^{10} - 17932q^8 + 21741q^6 - 24913q^4 + \\
& + 27008q^2 - 27736 + 27008q^{-2} - 24913q^{-4} + 21741q^{-6} - 17932q^{-8} + 13956q^{-10} - 10201q^{-12} + 6974q^{-14} - 4446q^{-16} + 2623q^{-18} - 1410q^{-20} + 684q^{-22} - \\
& -298q^{-24} + 111q^{-26} - 30q^{-28} + 4q^{-30}) + \frac{1}{A^8}(-6q^{30} + 40q^{28} - 140q^{26} + 372q^{24} - 850q^{22} + 1723q^{20} - 3143q^{18} + 5251q^{16} - 8121q^{14} + 11694q^{12} - 15804q^{10} + \\
& + 20148q^8 - 24233q^6 + 27575q^4 - 29822q^2 + 30632 - 29822q^{-2} + 27575q^{-4} - 24233q^{-6} + 20148q^{-8} - 15804q^{-10} + 11694q^{-12} - 8121q^{-14} + 5251q^{-16} - \\
& -3143q^{-18} + 1723q^{-20} - 850q^{-22} + 372q^{-24} - 140q^{-26} + 40q^{-28} - 6q^{-30}) + \frac{1}{A^{10}}(4q^{30} - 22q^{28} + 76q^{26} - 208q^{24} + 470q^{22} - 930q^{20} + 1676q^{18} - 2766q^{16} + \\
& + 4201q^{14} - 5990q^{12} + 8048q^{10} - 10160q^8 + 12146q^6 - 13825q^4 + 14904q^2 - 15260 + 14904q^{-2} - 13825q^{-4} + 12146q^{-6} - 10160q^{-8} + 8048q^{-10} - 5990q^{-12} + \\
& + 4201q^{-14} - 2766q^{-16} + 1676q^{-18} - 930q^{-20} + 470q^{-22} - 208q^{-24} + 76q^{-26} - 22q^{-28} + 4q^{-30}) + \frac{1}{A^{12}}(-q^{30} + 4q^{28} - 19q^{26} + 56q^{24} - 122q^{22} + 233q^{20} - \\
& -396q^{18} + 604q^{16} - 849q^{14} + 1103q^{12} - 1323q^{10} + 1519q^8 - 1660q^6 + 1734q^4 - 1775q^2 + 1801 - 1775q^{-2} + 1734q^{-4} - 1660q^{-6} + 1519q^{-8} - 1323q^{-10} + \\
& + 1103q^{-12} - 849q^{-14} + 604q^{-16} - 396q^{-18} + 233q^{-20} - 122q^{-22} + 56q^{-24} - 19q^{-26} + 4q^{-28} - q^{-30}) + \frac{1}{A^{14}}(3q^{26} - 7q^{24} + 18q^{22} - 30q^{20} + 30q^{18} - \\
& -17q^{16} - 13q^{14} + 96q^{12} - 239q^{10} + 396q^8 - 566q^6 + 742q^4 - 862q^2 + 892 - 862q^{-2} + 742q^{-4} - 566q^{-6} + 396q^{-8} - 239q^{-10} + 96q^{-12} - 13q^{-14} - 17q^{-16} + \\
& + 30q^{-18} - 30q^{-20} + 18q^{-22} - 7q^{-24} + 3q^{-26}) + \frac{1}{A^{16}}(-3q^{22} - q^{20} + 4q^{18} - 12q^{16} + 37q^{14} - 67q^{12} + 85q^{10} - 134q^8 + 176q^6 - 170q^4 + 194q^2 - 230 + \\
& + 194q^{-2} - 170q^{-4} + 176q^{-6} - 134q^{-8} + 85q^{-10} - 67q^{-12} + 37q^{-14} - 12q^{-16} + 4q^{-18} - q^{-20} - 3q^{-22}) + \frac{1}{A^{18}}(q^{18} + 7q^{16} - 5q^{14} + 6q^{12} - 18q^{10} + 15q^8 - \\
& -q^6 + 24q^4 - 17q^2 - 4 - 17q^{-2} + 24q^{-4} - q^{-6} + 15q^{-8} - 18q^{-10} + 6q^{-12} - 5q^{-14} + 7q^{-16} + q^{-18}) + \frac{1}{A^{20}}(-3q^{12} - 5q^{10} + 2q^8 + 2q^6 + 8q^4 - 10q^2 - 3 - \\
& -10q^{-2} + 8q^{-4} + 2q^{-6} + 2q^{-8} - 5q^{-10} - 3q^{-12}) + \frac{1}{A^{22}}(3q^6 + q^4 + q^2 - 4 + q^{-2} + q^{-4} + 3q^{-6}) - \frac{1}{A^{24}}]
\end{aligned}$$

⁴Evaluating $H_{[21]}$ for 5-strand closed braids looks currently unaffordable.

Note that a bit more compact is to tabulate not the polynomials themselves, but somewhat more concise data, polynomials $G_R(A, q)$, an abbreviation made possible by existence of differential expansions [44, 35, 36, 45, 46]

$$\begin{aligned}
H_{[1]} &= 1 + G_1 \cdot \{Aq\}\{A/q\}, \\
H_{[2]} &= 1 + [2]G_1 \cdot \{Aq^2\}\{A/q\} + G_2 \cdot \{Aq^3\}\{Aq^2\}\{A/q\}, \\
H_{[3]} &= 1 + [3]G_1 \cdot \{Aq^3\}\{A/q\} + [3]G_2 \cdot \{Aq^4\}\{Aq^3\}\{A/q\} + G_3 \cdot \{Aq^5\}\{Aq^4\}\{Aq^3\}\{A/q\}, \\
H_{[2,1]} &= 1 + G_1 \cdot \left(\{Aq^3\}\{A/q^3\} + \{Aq^2\}\{A\} + \{A\}\{A/q^2\} \right) + \\
&\quad + \{Aq^2\}\{A/q^2\} \cdot \left([3]\{A\} \cdot G_2(q=1) + \{Aq^3\}\{A/q^3\} \cdot G_3(q=1) + \{q\}^2 \cdot G_{21} \right)
\end{aligned} \tag{54}$$

Note that $G_1(A, -q^{-1}) = G_1(A, q)$, but G_2 and G_3 do not obligatory possess this symmetry, therefore, the decomposition of $H_{[2,1]}$ involves their values at $q = 1$. This choice of G_{21} does not look canonical and can deserve modification.

We remind that for the transposed Young diagram

$$H_{\tilde{R}}(A, q) = H_R(A, -q^{-1}) \tag{55}$$

and for the mirror knot

$$H_{\tilde{R}}^{\mathcal{K}}(A, q) = H_R^{\mathcal{K}}(-A^{-1}, q) \tag{56}$$

If the knot has zero *defect* [46], for example, belongs to the twist-knots family, then G_1 is independent of q , while G_2 and G_3 are further factorized: they are divisible by $\{A\}$ and $\{A\}\{Aq\}$ respectively, then G_{21} is also divisible by $\{A\}$ (this follows from (54), since at $A = 1$ ones gets the Alexander polynomial $Al_R(q)$, which celebrates the property $Al_R(q) = Al_{[1]}(q^{|R|})$ for any hook diagram R , [23]).

As an illustration of this kind decomposition, we write it for knot 8_1 which is the twist knot with twist=-3 and which is described by at least a 5-strand closed braid:

$$\begin{aligned}
G_1^{8_1} &= \frac{A^4 + A^2 + 1}{A^4}, \\
G_2^{8_1} &= \{A\} \frac{A^8 q^8 + A^6 q^8 + A^6 q^6 + A^4 q^8 + A^4 q^6 + A^4 q^4 + A^2 q^4 + A^2 q^2 + 1}{A^8 q^8}, \\
G_3^{8_1} &= \{A\}\{Aq\} A^{-12} q^{-24} \left(A^{12} q^{24} + A^{10} q^{24} + A^{10} q^{22} + A^{10} q^{20} + A^8 q^{24} + A^8 q^{22} + 2A^8 q^{20} + A^8 q^{18} + A^8 q^{16} + A^6 q^{20} + \right. \\
&\quad \left. + 2A^6 q^{18} + 2A^6 q^{16} + A^6 q^{14} + A^6 q^{12} + A^4 q^{16} + A^4 q^{14} + 2A^4 q^{12} + A^4 q^{10} + A^4 q^8 + A^2 q^8 + A^2 q^6 + A^2 q^4 + 1 \right), \\
G_{21}^{8_1} &= \{A\} A^{-13} q^{-12} \left(-7A^6 q^8 + A^4 q^8 - 4A^8 q^6 - 4A^6 q^6 + 2A^4 q^4 - A^6 q^2 - A^{10} q^{20} - 2A^8 q^{20} - A^{12} q^{18} - 2A^{10} q^{18} - \right. \\
&\quad - 4A^8 q^{18} - 4A^6 q^{18} - 2A^{10} q^{16} - 4A^8 q^{16} - 5A^8 q^{14} - 8A^6 q^{14} - 4A^8 q^{12} - 9A^6 q^{12} - 5A^8 q^{10} - 8A^6 q^{10} - 2A^{10} q^8 - \\
&\quad - 4A^8 q^8 - 2A^8 q^4 - 3A^6 q^4 - A^8 q^2 - 2A^{10} q^6 - A^{10} q^4 - 6A^4 q^{10} - A^4 q^6 + A^2 q^4 - 6A^4 q^{14} - 3A^2 q^{14} - 2A^4 q^{12} - \\
&\quad - 2A^2 q^{12} - 3A^2 q^{10} + A^{14} q^{14} + A^{14} q^{12} + 2A^{12} q^{12} + A^{14} q^{10} - A^{12} q^6 + A^4 - q^{14} - q^{12} - q^{10} - A^6 + A^2 q^{20} + \\
&\quad \left. + A^4 q^{24} - A^6 q^{24} - A^6 q^{22} - 3A^6 q^{20} + 2A^4 q^{20} - A^4 q^{18} + A^4 q^{16} - A^8 q^{22} - 7A^6 q^{16} \right)
\end{aligned}$$

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Appendix A

Here we list the remaining \mathcal{R} -matrices and mixing matrices that we calculated directly from highest weight vectors, see s.2:

$$\mathcal{R}_{[5,3,1]} = \begin{pmatrix} q^{-\mathcal{K}[4,2]} & & & & & \\ & q^{-\mathcal{K}[4,2]} & & & & \\ & & -q^{-\mathcal{K}[4,1,1]} & & & \\ & & & -q^{-\mathcal{K}[3,3]} & & \\ & & & & -q^{-\mathcal{K}[3,2,1]} & \\ & & & & & q^{-\mathcal{K}[3,2,1]} \end{pmatrix} = \begin{pmatrix} \frac{1}{q^5} & & & & & \\ & \frac{1}{q^5} & & & & \\ & & -\frac{1}{q^3} & & & \\ & & & -\frac{1}{q^3} & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix} \quad (57)$$

$$\mathcal{U}_{[5,3,1]} = \begin{pmatrix} -\frac{[9]+[5]+2[3]-1}{[5][2]\alpha_1} & -\frac{\alpha_0}{\alpha_1} \sqrt{\frac{[7][3]}{[5]}} & \frac{\alpha_0^2}{\sqrt{[5][3]\alpha_1}} & \frac{\sqrt{\alpha_1}}{[2]\sqrt{[5][3]}} & X_{15} & X_{16} \\ -\frac{\alpha_0}{\alpha_1} \sqrt{\frac{[7][3]}{[5]}} & \frac{[3]^2}{[2]\alpha_1} & \frac{\sqrt{[7]}}{[2]\sqrt{\alpha_1}} & 0 & X_{25} & X_{26} \\ \frac{\alpha_0^2}{\sqrt{[5][3]\alpha_1}} & \frac{\sqrt{[7]}}{[2]\sqrt{\alpha_1}} & \frac{1}{[2][3]} & \frac{\alpha_0}{[3]} & X_{35} & X_{36} \\ \frac{\sqrt{\alpha_1}}{[2]\sqrt{[5][3]}} & 0 & \frac{\alpha_0}{[3]} & \frac{1}{[2][3]} & X_{45} & X_{46} \\ X_{15} & X_{25} & X_{35} & X_{45} & X_{55} & X_{56} \\ X_{16} & X_{26} & X_{36} & X_{46} & X_{56} & X_{66} \end{pmatrix} \quad (58)$$

with $\alpha_1 = [7] + [5] + 1$, $\alpha_0 = [3] - 1$.

$$\begin{aligned} X_{15} &= -1/2 \frac{\sqrt{q^4+1}\sqrt{2}(q^6-q^5+q^4-q^3+q^2-q+1)\sqrt{q^2-q+1}\sqrt{q^4+q^2+1+q^3+q}}{\sqrt{q^2+q+1}(q^4+q^2+1-q^3-q)q^3\sqrt{\alpha_1}} \\ X_{16} &= 1/2 \frac{\sqrt{q^4+1}\sqrt{2}(q^6+q^5+q^4+q^3+q^2+q+1)\sqrt{q^2+q+1}\sqrt{q^4+q^2+1-q^3-q}}{\sqrt{q^2-q+1}(q^4+q^2+1+q^3+q)q^3\sqrt{\alpha_1}} \\ X_{25} &= 1/2 \frac{\sqrt{q^4+1}\sqrt{q^6+q^5+q^4+q^3+q^2+q+1}\sqrt{q^6-q^5+q^4-q^3+q^2-q+1}\sqrt{2}}{\sqrt{q^4+q^2+1-q^3-q}q^3\sqrt{\alpha_1}} \\ X_{26} &= 1/2 \frac{\sqrt{q^4+1}\sqrt{q^6+q^5+q^4+q^3+q^2+q+1}\sqrt{q^6-q^5+q^4-q^3+q^2-q+1}\sqrt{2}}{\sqrt{q^4+q^2+1+q^3+q}q^3\sqrt{\alpha_1}} \\ X_{35} &= -1/2 \frac{q\sqrt{2}\sqrt{q^4+1}}{\sqrt{q^4+q^2+1-q^3-q}(q^2+q+1)}, \quad X_{36} = 1/2 \frac{q\sqrt{2}\sqrt{q^4+1}}{\sqrt{q^4+q^2+1+q^3+q}(q^2-q+1)} \\ X_{45} &= 1/2 \frac{q\sqrt{2}\sqrt{q^4+1}}{\sqrt{q^4+q^2+1-q^3-q}(q^2+q+1)}, \quad X_{46} = 1/2 \frac{q\sqrt{2}\sqrt{q^4+1}}{\sqrt{q^4+q^2+1+q^3+q}(q^2-q+1)} \\ X_{55} &= -\frac{q^3}{(q^2+q+1)(q^4+q^2+1-q^3-q)}, \quad X_{56} = 0 \\ X_{65} &= 0, \quad X_{66} = -\frac{q^3}{(q^2-q+1)(q^4+q^2+1+q^3+q)} \end{aligned}$$

$$\mathcal{R}_{[5,2,1,1]} = \begin{pmatrix} q^{-\mathcal{K}[4,2]} & & & & & \\ & -q^{-\mathcal{K}[4,1,1]} & & & & \\ & & -q^{-\mathcal{K}[4,1,1]} & & & \\ & & & -q^{-\mathcal{K}[3,2,1]} & & \\ & & & & q^{-\mathcal{K}[3,2,1]} & \\ & & & & & q^{-\mathcal{K}[3,1,1,1]} \end{pmatrix} = \begin{pmatrix} \frac{1}{q^5} & & & & & \\ & -\frac{1}{q^3} & & & & \\ & & -\frac{1}{q^3} & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & q^3 \end{pmatrix} \quad (59)$$

$$\mathcal{U}_{[5,2,1,1]} = \begin{pmatrix} \frac{[3]}{[5][2]} & \frac{1}{[2]} \sqrt{\frac{[5]-[3]}{[5]\alpha_0}} & -\sqrt{\frac{[2]}{[6]\alpha_0}} & Y_{14} & Y_{15} & ([3]-1) \sqrt{\frac{([5]-[3])[2]}{[6][5]}} \\ \frac{1}{[2]} \sqrt{\frac{[5]-[3]}{[5]\alpha_0}} & \frac{1}{[2]} & 0 & Y_{24} & Y_{25} & \frac{1}{[3]} \sqrt{\frac{[6]}{[2]\alpha_0}} \\ -\sqrt{\frac{[2]}{[6]\alpha_0}} & 0 & -\frac{1}{[6]} & Y_{34} & Y_{35} & \frac{1}{[6]} \sqrt{\frac{[5]([5]-[3])}{[3]-1}} \\ Y_{14} & Y_{24} & Y_{34} & Y_{44} & Y_{45} & Y_{46} \\ Y_{15} & Y_{25} & Y_{35} & Y_{45} & Y_{55} & Y_{56} \\ ([3]-1) \sqrt{\frac{([5]-[3])[2]}{[6][5]}} & \frac{1}{[3]} \sqrt{\frac{[6]}{[2]\alpha_0}} & \frac{1}{[6]} \sqrt{\frac{[5]([5]-[3])}{[3]-1}} & Y_{64} & Y_{65} & \frac{1}{[6]} \end{pmatrix} \quad (60)$$

$$\begin{aligned} Y_{14} &= 1/2 \frac{\sqrt{q^8+1}\sqrt{q^4+1}\sqrt{2}q}{\sqrt{q^4+q^2+1+q^3+q}\sqrt{q^4+q^2+1-q^3-q}\sqrt{q^2+q+1}\sqrt{q^2-q+1}} \\ Y_{15} &= 1/2 \frac{\sqrt{q^8+1}\sqrt{q^4+1}\sqrt{2}q}{(q^4+q^2+1+q^3+q)\sqrt{q^4+q^2+1-q^3-q}\sqrt{q^2+q+1}\sqrt{q^2-q+1}} \\ Y_{24} &= 1/2 \frac{(q-1)^2 \sqrt{q^2+q+1}\sqrt{2}}{\sqrt{q^2-q+1}\sqrt{q^4+q^2+1-q^3-q}}, \quad Y_{25} = 1/2 \frac{(1+q)^2 \sqrt{q^2-q+1}\sqrt{2}}{\sqrt{q^2+q+1}\sqrt{q^4+q^2+1+q^3+q}} \\ Y_{34} &= -1/2 \frac{\sqrt{q^8+1}\sqrt{2}\sqrt{q^4+q^2+1+q^3+q}}{(q^2-q+1)(q^2+q+1)\sqrt{q^4-q^2+1}}, \quad Y_{35} = 1/2 \frac{\sqrt{q^8+1}\sqrt{2}\sqrt{q^4+q^2+1-q^3-q}}{(q^2-q+1)(q^2+q+1)\sqrt{q^4-q^2+1}} \\ Y_{44} &= -1/2 \frac{q(q^6-2q^5+3q^4-2q^3+3q^2-2q+1)}{(q^2-q+1)(q^2+q+1)(q^4+q^2+1-q^3-q)} \\ Y_{45} &= -1/2 \frac{(q^2+1)(q^4+1)q}{(q^2-q+1)(q^2+q+1)\sqrt{q^4+q^2+1+q^3+q}\sqrt{q^4+q^2+1-q^3-q}} \\ Y_{55} &= -1/2 \frac{q(q^6+2q^5+3q^4+2q^3+3q^2+2q+1)}{(q^4+q^2+1+q^3+q)(q^2-q+1)(q^2+q+1)} \\ Y_{64} &= -1/2 \frac{\sqrt{2}\sqrt{q^4+1}q^3}{(q^2-q+1)(q^2+q+1)\sqrt{q^4+q^2+1-q^3-q}\sqrt{q^4-q^2+1}} \\ Y_{65} &= 1/2 \frac{\sqrt{2}\sqrt{q^4+1}q^3}{(q^2-q+1)(q^2+q+1)\sqrt{q^4+q^2+1+q^3+q}\sqrt{q^4-q^2+1}} \end{aligned}$$

$$\mathcal{R}_{[4,3,2]} = \begin{pmatrix} q^{-\mathcal{K}_{[4,2]}} & & & & & & & \\ & -q^{-\mathcal{K}_{[4,1,1]}} & & & & & & \\ & & -q^{-\mathcal{K}_{[3,3]}} & & & & & \\ & & & -q^{-\mathcal{K}_{[3,2,1]}} & & & & \\ & & & & -q^{-\mathcal{K}_{[3,2,1]}} & & & \\ & & & & & q^{-\mathcal{K}_{[3,2,1]}} & & \\ & & & & & & q^{-\mathcal{K}_{[3,2,1]}} & \\ & & & & & & & q^{-\mathcal{K}_{[2,2,2]}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{q^5} & & & & & & & \\ & -\frac{1}{q^3} & & & & & & \\ & & -\frac{1}{q^3} & & & & & \\ & & & -1 & & & & \\ & & & & -1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & q^3 \end{pmatrix} \quad (61)$$

$$\mathcal{U}_{[4,3,2]} = \begin{pmatrix} \frac{[5]+2}{[5][2]^2\alpha_0} & \frac{1}{[2]^2}\sqrt{\frac{[6]}{[5][2]\alpha_0}} & \frac{1}{[2]^2}\sqrt{\frac{[6]}{[5][2]\alpha_0}} & Z_{14} & 0 & 0 & Z_{17} & \frac{[3]}{[2]^2\alpha_0}\sqrt{\frac{[6]}{[2]}} \\ \frac{1}{[2]^2}\sqrt{\frac{[6]}{[5][2]\alpha_0}} & \frac{1}{[2]^2} & \frac{1}{[2]^2} & Z_{24} & \frac{1}{2} & \frac{1}{2} & Z_{27} & -\frac{1}{[2]^2}\sqrt{\frac{[5]}{\alpha_0}} \\ \frac{1}{[2]^2}\sqrt{\frac{[6]}{[5][2]\alpha_0}} & \frac{1}{[2]^2} & \frac{1}{[2]^2} & Z_{34} & -\frac{1}{2} & -\frac{1}{2} & Z_{37} & -\frac{1}{[2]^2}\sqrt{\frac{[5]}{\alpha_0}} \\ Z_{14} & Z_{24} & Z_{34} & Z_{44} & 0 & 0 & Z_{74} & Z_{84} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ Z_{17} & Z_{27} & Z_{37} & Z_{47} & 0 & 0 & Z_{77} & Z_{87} \\ \frac{[3]}{[2]^2\alpha_0}\sqrt{\frac{[6]}{[2]}} & -\frac{1}{[2]^2}\sqrt{\frac{[5]}{\alpha_0}} & -\frac{1}{[2]^2}\sqrt{\frac{[5]}{\alpha_0}} & Z_{84} & 0 & 0 & Z_{87} & \frac{1}{[2]^2\alpha_0} \end{pmatrix} \quad (62)$$

$$\begin{aligned} Z_{14} &= \frac{q\sqrt{q^2+q+1}\sqrt{q^4-q^2+1}(q^2-q+1)^{3/2}}{\sqrt{q^4+1}(q^2+1)(q^4+q^2+1-q^3-q)} & Z_{17} &= -\frac{q\sqrt{q^2-q+1}\sqrt{q^4-q^2+1}(q^2+q+1)^{3/2}}{\sqrt{q^4+1}(q^2+1)(q^4+q^2+1+q^3+q)} \\ Z_{24} &= 1/2 \frac{\sqrt{q^4+q^2+1+q^3+q}(q-1)^2}{\sqrt{q^4+q^2+1-q^3-q}(q^2+1)} & Z_{27} &= -1/2 \frac{\sqrt{q^4+q^2+1-q^3-q}(1+q)^2}{\sqrt{q^4+q^2+1+q^3+q}(q^2+1)} \\ Z_{34} &= 1/2 \frac{\sqrt{q^4+q^2+1+q^3+q}(q-1)^2}{\sqrt{q^4+q^2+1-q^3-q}(q^2+1)} & Z_{37} &= -1/2 \frac{\sqrt{q^4+q^2+1-q^3-q}(1+q)^2}{\sqrt{q^4+q^2+1+q^3+q}(q^2+1)} \\ Z_{44} &= 1/2 \frac{q^4+q^3-3q^2+q+1}{q^4+q^2+1-q^3-q} & Z_{47} &= Z_{74} = \frac{1}{2} \\ Z_{54} &= Z_{57} = Z_{64} = Z_{67} = 0 & Z_{77} &= 1/2 \frac{q^4-q^3-3q^2-q+1}{q^4+q^2+1+q^3+q} \\ Z_{84} &= -\frac{q^2}{\sqrt{q^4+1}(q^2+1)} & Z_{87} &= -\frac{q^2}{\sqrt{q^4+1}(q^2+1)} \end{aligned}$$

$$\mathcal{R}_{[4,3,1,1]} = \begin{pmatrix} q^{-\mathcal{K}_{[4,2]}} & & & & & & & \\ & -q^{-\mathcal{K}_{[4,1,1]}} & & & & & & \\ & & -q^{-\mathcal{K}_{[3,3]}} & & & & & \\ & & & -q^{-\mathcal{K}_{[3,2,1]}} & & & & \\ & & & & -q^{-\mathcal{K}_{[3,2,1]}} & & & \\ & & & & & q^{-\mathcal{K}_{[3,2,1]}} & & \\ & & & & & & q^{-\mathcal{K}_{[3,2,1]}} & \\ & & & & & & & q^{-\mathcal{K}_{[3,1,1,1]}} \\ & & & & & & & & -q^{-\mathcal{K}_{[2,2,1,1]}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{q^5} & & & & & & & \\ & -\frac{1}{q^3} & & & & & & \\ & & -\frac{1}{q^3} & & & & & \\ & & & -1 & & & & \\ & & & & -1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & q^3 \\ & & & & & & & & -q^5 \end{pmatrix} \quad (63)$$

$$\mathcal{U}_{[4,3,1,1]} = \begin{pmatrix} \frac{[3]}{[5][2]^2} & \frac{1}{[2]} \sqrt{\frac{\alpha_0}{[6][2]}} & \frac{[3]}{[2]^2} \sqrt{\frac{1}{[5]\alpha_0}} & W_{1,4} & W_{1,5} & W_{1,6} & W_{1,7} & \frac{1}{[2]} \sqrt{\frac{[7]}{[6][2]}} & \frac{[3]}{[2]^2} \sqrt{\frac{[7]}{[5]\alpha_0}} \\ \frac{1}{[2]} \sqrt{\frac{\alpha_0}{[6][2]}} & \frac{\alpha_2}{[6][2]} & \frac{1}{[2]} \sqrt{\frac{[5]}{[6][2]}} & W_{2,4} & W_{2,5} & W_{2,6} & W_{2,7} & -\frac{1}{[6][2]} \sqrt{[7]\alpha_0} & -\frac{[3]}{[2]} \sqrt{\frac{[7]}{[6][5][2]}} \\ \frac{[3]}{[2]^2} \sqrt{\frac{1}{[5]\alpha_0}} & \frac{1}{[2]} \sqrt{\frac{[5]}{[6][2]}} & -\frac{1}{[2]^2 \alpha_0} & W_{3,4} & W_{3,5} & W_{3,6} & W_{3,7} & \frac{1}{[2]} \sqrt{\frac{[7][5]}{[6][2]\alpha_0}} & -\frac{1}{[2]^2 \alpha_0} \sqrt{[7]} \\ W_{1,4} & W_{2,4} & W_{3,4} & W_{4,4} & W_{4,5} & W_{4,6} & W_{4,7} & W_{8,4} & W_{9,4} \\ W_{1,5} & W_{2,5} & W_{3,5} & W_{5,4} & W_{5,5} & W_{5,6} & W_{5,7} & W_{8,5} & W_{9,5} \\ W_{1,6} & W_{2,6} & W_{3,6} & W_{6,4} & W_{6,5} & W_{6,6} & W_{6,7} & W_{8,6} & W_{9,6} \\ W_{1,7} & W_{2,7} & W_{3,7} & W_{7,4} & W_{7,5} & W_{7,6} & W_{7,7} & W_{8,7} & W_{9,7} \\ \frac{1}{[2]} \sqrt{\frac{[7]}{[6][2]}} & -\frac{1}{[6][2]} \sqrt{[7]\alpha_0} & \frac{1}{[2]} \sqrt{\frac{[7][5]}{[6][2]\alpha_0}} & W_{8,4} & W_{8,5} & W_{8,6} & W_{8,7} & \frac{1}{[6][2]} & \frac{[3]}{[2]^2} \sqrt{\frac{[2]}{[6][5]\alpha_0}} \\ \frac{[3]}{[2]^2} \sqrt{\frac{[7]}{[5]\alpha_0}} & -\frac{[3]}{[2]} \sqrt{\frac{[7]}{[6][5][2]}} & -\frac{1}{[2]^2 \alpha_0} \sqrt{[7]} & W_{9,4} & W_{9,5} & W_{9,6} & W_{9,7} & \frac{[3]}{[2]^2} \sqrt{\frac{[2]}{[6][5]\alpha_0}} & -\frac{[3]}{[5][2]^2 \alpha_0} \end{pmatrix} \quad (64)$$

$\alpha_2 = [5] - [3]$

$$\begin{aligned} W_{1,4} &= 1/2 \frac{\sqrt{2}q^2 \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1}}{\sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^2 - q + 1} (q^2 + 1) \sqrt{q^2 + q + 1} (q^4 + q^2 + 1 - q^3 - q)} \\ W_{2,4} &= 1/2 \frac{\sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1} q \sqrt{q^4 + 1} \sqrt{2}}{\sqrt{q^4 + q^2 + 1 + q^3 + q} (q^2 + 1) (q^2 - q + 1) (q^2 + q + 1) \sqrt{q^4 - q^2 + 1}} \\ W_{3,4} &= 1/2 \frac{\sqrt{2}q \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1} \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1}}{\sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^2 + q + 1} \sqrt{q^2 - q + 1} (q^2 + 1) \sqrt{q^4 + 1}} \\ W_{4,4} &= 1/2 \frac{q^{12} - q^{11} + q^{10} - q^9 + q^8 - 2q^7 - 2q^5 + q^4 - q^3 + q^2 - q + 1}{(q^2 - q + 1) (q^2 + q + 1) (q^4 + q^2 + 1 + q^3 + q) (q^4 + q^2 + 1 - q^3 - q)} \\ W_{5,4} &= -1/2 \frac{(q^4 + q^3 - q^2 + q + 1) \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1}}{(q^4 + q^2 + 1 + q^3 + q) \sqrt{q^2 - q + 1} \sqrt{q^2 + q + 1} (q^4 + q^2 + 1 - q^3 - q)} \\ W_{6,4} &= 1/2 \frac{(q^{10} - 2q^9 + 2q^8 - 3q^7 + 2q^6 - 3q^5 + 2q^4 - 3q^3 + 2q^2 - 2q + 1) \sqrt{q^2 + q + 1}}{\sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^2 - q + 1} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\ W_{7,4} &= -1/2 \frac{(q^2 + 1) (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)^{3/2} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1}}{(q^2 - q + 1) (q^2 + q + 1) \sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\ W_{8,4} &= -1/2 \frac{\sqrt{2}q^5}{\sqrt{q^4 + q^2 + 1 + q^3 + q} (q^2 + 1) (q^2 - q + 1) (q^2 + q + 1) \sqrt{q^4 - q^2 + 1}} \\ W_{9,4} &= -1/2 \frac{(q^2 + q + 1)^{3/2} \sqrt{2}q^3 \sqrt{q^2 - q + 1}}{(q^4 + q^2 + 1 + q^3 + q) (q^2 + 1) \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^4 + 1}} \\ W_{1,5} &= 1/2 \frac{(q^6 - q^5 + q^4 - q^3 + q^2 - q + 1) q \sqrt{2}}{(q^4 + q^2 + 1 - q^3 - q) \sqrt{q^4 + q^2 + 1 + q^3 + q} (q^2 + 1)} \\ W_{2,5} &= 1/2 \frac{(q^6 + q^3 + 1) \sqrt{q^4 + 1} \sqrt{2}}{\sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^2 - q + 1} (q^2 + 1) \sqrt{q^2 + q + 1} \sqrt{q^4 - q^2 + 1}} \\ W_{3,5} &= -1/2 \frac{(q^2 - q + 1) q^2 \sqrt{2}}{\sqrt{q^4 + q^2 + 1 - q^3 - q} (q^2 + 1) \sqrt{q^4 + 1}} \\ W_{4,5} &= -1/2 \frac{(q^4 + q^3 - q^2 + q + 1) \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1}}{(q^4 + q^2 + 1 + q^3 + q) \sqrt{q^2 - q + 1} \sqrt{q^2 + q + 1} (q^4 + q^2 + 1 - q^3 - q)} \\ W_{5,5} &= \frac{(q^6 + q^3 + 1) q}{(q^4 + q^2 + 1 + q^3 + q) (q^4 + q^2 + 1 - q^3 - q)} \\ W_{6,5} &= -1/2 \frac{\sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1} (q^2 + 1) q}{\sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\ W_{7,5} &= -1/2 \frac{(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) (q^6 - q^5 + q^4 - q^3 + q^2 - q + 1)}{\sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^2 + q + 1} \sqrt{q^2 - q + 1} \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\ W_{8,5} &= 1/2 \frac{\sqrt{2}q \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1} \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1}}{\sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^2 - q + 1} (q^2 + 1) \sqrt{q^2 + q + 1} \sqrt{q^4 - q^2 + 1}} \\ W_{9,5} &= -1/2 \frac{\sqrt{2}q^2 \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1}}{(q^4 + q^2 + 1 + q^3 + q) (q^2 + 1) \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^4 + 1}} \end{aligned}$$

$$\begin{aligned}
W_{1,6} &= 1/2 \frac{q(q^2 - q + 1)(q^2 + q + 1)^2 \sqrt{2} \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1}}{(q^4 + q^2 + 1 + q^3 + q)(q^2 + 1) \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{2,6} &= 1/2 \frac{\sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1} \sqrt{q^4 + 1} \sqrt{q^2 + q + 1} \sqrt{2}}{\sqrt{q^4 - q^2 + 1} \sqrt{q^2 - q + 1} (q^2 + 1) \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{3,6} &= -1/2 \frac{\sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1} q^3 \sqrt{2}}{\sqrt{q^4 + 1} \sqrt{q^4 + q^2 + 1 + q^3 + q} (q^2 + 1) \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{4,6} &= 1/2 \frac{(q^{10} - 2q^9 + 2q^8 - 3q^7 + 2q^6 - 3q^5 + 2q^4 - 3q^3 + 2q^2 - 2q + 1) \sqrt{q^2 + q + 1}}{\sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^2 - q + 1} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{5,6} &= -1/2 \frac{\sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1} (q^2 + 1) q}{\sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{6,6} &= -1/2 \frac{q^2 (1 - 2q^3 - 4q^7 - 4q^5 - q^4 - 2q^9 - q^8 - 2q^6 + q^{12})}{(q^4 + q^2 + 1 + q^3 + q)(q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1)} \\
W_{7,6} &= 1/2 \frac{(q^6 - q^5 + q^4 + q^3 + q^2 - q + 1) \sqrt{q^2 + q + 1} \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1}}{(q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1) \sqrt{q^2 - q + 1}} \\
W_{8,6} &= -1/2 \frac{\sqrt{q^4 + q^2 + 1 - q^3 - q} (q^6 + q^3 + 1) q \sqrt{2} \sqrt{q^2 + q + 1}}{\sqrt{q^4 - q^2 + 1} \sqrt{q^2 - q + 1} (q^2 + 1) \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{9,6} &= 1/2 \frac{\sqrt{2} q^2 (q^8 + q^7 + 2q^6 + q^5 + 3q^4 + q^3 + 2q^2 + q + 1)}{\sqrt{q^4 + 1} \sqrt{q^4 + q^2 + 1 + q^3 + q} (q^2 + 1) \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{1,7} &= -1/2 \frac{\sqrt{2} q^3 (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)}{\sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1} (q^2 + 1) \sqrt{q^2 - q + 1} \sqrt{q^2 + q + 1} \sqrt{q^4 + q^2 + 1 - q^3 - q}} \\
W_{2,7} &= -1/2 \frac{(q^8 - q^7 + q^6 - 2q^5 + q^4 - 2q^3 + q^2 - q + 1) q^3 \sqrt{q^4 + 1} \sqrt{2}}{\sqrt{q^4 - q^2 + 1} (q^2 + q + 1) (q^2 - q + 1) (q^2 + 1) \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{3,7} &= 1/2 \frac{(q^8 - q^7 + 2q^6 - q^5 + 3q^4 - q^3 + 2q^2 - q + 1) q \sqrt{2} \sqrt{q^4 + q^2 + 1 + q^3 + q}}{\sqrt{q^2 + q + 1} \sqrt{q^2 - q + 1} \sqrt{q^4 + 1} (q^2 + 1) \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{4,7} &= -1/2 \frac{(q^2 + 1) (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)^{3/2} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1}}{(q^2 - q + 1) (q^2 + q + 1) \sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{5,7} &= -1/2 \frac{(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) (q^6 - q^5 + q^4 - q^3 + q^2 - q + 1)}{\sqrt{q^4 + q^2 + 1 + q^3 + q} \sqrt{q^2 + q + 1} \sqrt{q^2 - q + 1} \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{6,7} &= 1/2 \frac{(q^6 - q^5 + q^4 + q^3 + q^2 - q + 1) \sqrt{q^2 + q + 1} \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1}}{(q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1) \sqrt{q^2 - q + 1}} \\
W_{7,7} &= 1/2 \frac{[11] - 2[10] + 4[9] - 6[8] + 10[7] - 12[6] + 14[5] - 13[4] + 15[3] - 11[2] + 6}{(q^4 + q^2 + 1 - q^3 - q)(q^2 + q + 1)(q^2 - q + 1)(q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1)} \\
W_{8,7} &= 1/2 \frac{\sqrt{2} q^2 \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1} (q^6 + 2q^4 - q^3 + 2q^2 + 1)}{\sqrt{q^4 - q^2 + 1} (q^2 + q + 1) (q^2 - q + 1) (q^2 + 1) \sqrt{q^4 + q^2 + 1 - q^3 - q} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}} \\
W_{9,7} &= -1/2 \frac{q^4 \sqrt{2} \sqrt{q^6 + q^5 + q^4 + q^3 + q^2 + q + 1} \sqrt{q^6 - q^5 + q^4 - q^3 + q^2 - q + 1} \sqrt{q^2 + q + 1} \sqrt{q^2 - q + 1}}{(q^4 + q^2 + 1 - q^3 - q) \sqrt{q^4 + q^2 + 1 + q^3 + q} (q^2 + 1) \sqrt{q^4 + 1} \sqrt{q^{12} + 2q^{10} + 4q^8 + 5q^6 + 4q^4 + 2q^2 + 1}}
\end{aligned}$$

For transposed diagrams

$$\mathcal{R}_{\tilde{Q}}(q) = \mathcal{R}_Q(-q^{-1}), \quad \mathcal{U}_{\tilde{Q}}(q) = \mathcal{U}_Q(-q^{-1}) \quad (65)$$

Minus sign is important: for example,

$$\mathcal{R}_{[2,2,2,1,1,1]} = -q^{\mathcal{R}_{[2,2,1,1]}} = -q^5 = \mathcal{R}_{[6,3]}(-q^{-1}) \quad (66)$$

and

$$\mathcal{R}_{[4,1,1,1,1,1]} = q^{\mathcal{R}_{[3,1,1]}} = q^3 = \mathcal{R}_{[6,1,1]}(-q^{-1}) \quad (67)$$

The list of transpositions:

matrix size	\tilde{Q}	Q
1	$[4, 1, 1, 1, 1]$ $[2, 2, 2, 1, 1]$	$[6, 1, 1, 1]$ $[6, 3]$
2	$[3, 2, 1, 1, 1]$ $[2, 2, 2, 2, 1]$ $[5, 1, 1, 1, 1]$ $[3, 3, 3]$	$[6, 2, 1]$ $[5, 4]$ symmetric symmetric
4	$[3, 3, 1, 1, 1]$ $[3, 2, 2, 2]$	$[5, 2, 2]$ $[4, 4, 1]$
6	$[4, 2, 1, 1, 1]$ $[3, 2, 2, 1, 1]$	$[5, 2, 1, 1]$ $[5, 3, 1]$
8	$[3, 3, 2, 1]$	$[4, 3, 2]$
9	$[4, 2, 2, 1]$	$[4, 3, 1, 1]$

(68)

Appendix B. Matrices T, \bar{T} and S, \bar{S}

In this Appendix we list the matrices T, \bar{T} and S, \bar{S} in representation $[2, 1]$ necessary for evaluating the $[2, 1]$ -colored HOMFLY polynomials through the formulas of s.4, see [28]. Due to participating the conjugated representation $[2, 1]$, all the quantities depend on the group $SU_q(N)$.

In this case there are seven different items X in the decomposition

$$l[2, 1] \otimes \underbrace{[2, 1]}_{[2^{N-2}, 1]} = [4, 3, 2^{N-4}, 1] \oplus [4, 2^{N-2}] \oplus [4, 2^{N-3}, 1, 1] \oplus [3, 3, 2^{N-3}] \oplus [3, 3, 2^{N-4}, 1, 1] \oplus \underline{2 \cdot [3, 2^{N-2}, 1]} \oplus \underbrace{[2^N]}_{singlet} \quad (69)$$

Only one (underlined) item $[3, 2^{N-2}, 1] = [2, 1^{N-2}] = Adj$ enters (69) with non-trivial multiplicity $m_{Adj} = 2$, however this makes the Racah matrices 10×10 , since $\sum_X m_X^2 = 6 \cdot 1 + 1 \cdot 2^2 = 10$.

The corresponding dimensions and eigenvalues are

X	d_X	t_X
$l[2^N]$	1	1
$l[4, 2^{N-3}, 1, 1]$	$\frac{[N-2][N-1][N+1][N+2]}{[2]^2}$	$-A^2$
$l[3, 3, 2^{N-3}]$	$\frac{[N-2][N-1][N+1][N+2]}{[2]^2}$	$-A^2$
$l[3, 3, 2^{N-4}, 1, 1]$	$\frac{[N-3][N]^2[N+1]}{[2]^2}$	$-q^{-2}A^2$
$l[4, 3, 2^{N-4}, 1]$	$\frac{[N-3][N-1]^2[N+1]^2[N+3]}{[3]^2}$	A^3
$l[4, 2^{N-2}]$	$\frac{[N-1][N]^2[N+3]}{[2]^2}$	q^2A^2
$\underline{l[3, 2^{N-2}, 1]}$	$\underline{[N-1][N+1]}$	$\underline{\pm A}$

(70)

Similarly in order to construct matrix S , one also needs the expansion (26) and

X	d_X	t_X
$l[2, 2, 2]$	$\frac{[N-2][N-1]^2[N]^2[N+1]}{[2]^2[3]^2[4]}$	$q^{-3}A^{-3}$
$l[4, 1, 1]$	$\frac{[N-2][N-1][N][N+1][N+2][N+3]}{[2]^2[3][6]}$	$-q^3A^{-3}$
$l[3, 3]$	$\frac{[N-1][N]^2[N+1]^2[N+2]}{[2]^2[3]^2[4]}$	$-q^3A^{-3}$
$l[2, 2, 1, 1]$	$\frac{[N-3][N-2][N-1][N]^2[N+1]}{[2]^2[4][5]}$	$-q^{-5}A^{-3}$
$l[3, 1, 1, 1]$	$\frac{[N-3][N-2][N-1][N][N+1][N+2]}{[2]^2[3][6]}$	$q^{-3}A^{-3}$
$l[4, 2]$	$\frac{[N-1][N]^2[N+1][N+2][N+3]}{[2]^2[4][5]}$	q^5A^{-3}
<u>$l[3, 2, 1]$</u>	<u>$\frac{[N-2][N-1][N]^2[N+1][N+2]}{[3]^2[5]}$</u>	<u>$\pm A^{-3}$</u>

(71)

Here we again encounter one item with multiplicity two. The diagonal matrices \bar{T} and T are read off from the last columns of (70) and (71):

$$\bar{T} = \left(\begin{array}{cccccc|cccc} 1 & & & & & & & & & \\ & -A^2 & & & & & & & & \\ & & -A^2 & & & & & & & \\ & & & -q^{-2}A^2 & & & & & & \\ & & & & A^3 & & & & & \\ & & & & & q^2A^2 & & & & \\ \hline & & & & & & A & & & \\ & & & & & & & -A & & \\ & & & & & & & & A & \\ & & & & & & & & & -A \end{array} \right) \quad (72)$$

and

$$T = \frac{1}{A^3} \left(\begin{array}{cccccc|cccc} q^{-3} & & & & & & & & & \\ & -q^3 & & & & & & & & \\ & & -q^3 & & & & & & & \\ & & & -q^3 & & & & & & \\ & & & & -q^{-5} & & & & & \\ & & & & & q^{-3} & & & & \\ & & & & & & q^5 & & & \\ \hline & & & & & & & 1 & & \\ & & & & & & & & -1 & \\ & & & & & & & & & -1 \\ & & & & & & & & & 1 \end{array} \right) \quad (73)$$

while the matrices \bar{S} and S are respectively

$\frac{[3]}{D_{-1}D_0D_1}$	$\frac{[3]}{[2]D_0}\sqrt{\frac{D_2D_{-2}}{D_1D_{-1}}}$	$\frac{[3]}{[2]D_0}\sqrt{\frac{D_2D_{-2}}{D_1D_{-1}}}$	$\frac{[3]}{[2]D_{-1}}\sqrt{\frac{D_{-3}}{D_1}}$	$\frac{\sqrt{D_3D_{-3}}}{D_0}$	$\frac{[3]}{[2]D_1}\sqrt{\frac{D_3}{D_{-1}}}$	$\frac{[3]}{D_0}\frac{1}{\sqrt{D_1D_{-1}}}$	$\frac{[3]}{D_0}\frac{1}{\sqrt{D_1D_{-1}}}$	0	0
$\frac{[3]}{[2]D_0}\sqrt{\frac{D_2D_{-2}}{D_1D_{-1}}}$	$\frac{[3]}{[2]^2D_0}$	$\frac{[3]}{[2]^2D_0}$	$-\frac{[3]}{[2]^2D_0}\sqrt{\frac{D_{-3}D_2}{D_{-2}D_{-1}}}$	0	$-\frac{[3]}{[2]^2D_0}\sqrt{\frac{D_3D_{-2}}{D_2D_1}}$	$\bar{S}_{[2,7]}$	$-\frac{\sqrt{D_2D_{-2}}}{[2]D_0}$	$\frac{D_{-1}}{[2]D_0}$	$\frac{D_1}{[2]D_0}$
$\frac{[3]}{[2]D_0}\sqrt{\frac{D_2D_{-2}}{D_1D_{-1}}}$	$\frac{[3]}{[2]^2D_0}$	$\frac{[3]}{[2]^2D_0}$	$-\frac{[3]}{[2]^2D_0}\sqrt{\frac{D_{-3}D_2}{D_{-1}D_{-2}}}$	0	$-\frac{[3]}{[2]^2D_0}\sqrt{\frac{D_3D_{-2}}{D_1D_2}}$	$\bar{S}_{[2,7]}$	$-\frac{\sqrt{D_2D_{-2}}}{[2]D_0}$	$-\frac{D_1}{[2]D_0}$	$-\frac{D_{-1}}{[2]D_0}$
$\frac{[3]}{[2]D_{-1}}\sqrt{\frac{D_3}{D_1}}$	$-\frac{[3]}{[2]^2D_0}\sqrt{\frac{D_{-3}D_2}{D_{-2}D_{-1}}}$	$-\frac{[3]}{[2]^2D_0}\sqrt{\frac{D_{-3}D_2}{D_{-1}D_{-2}}}$	$\bar{S}_{[4,4]}$	$\frac{[3]\sqrt{D_1D_3}}{D_0D_2D_{-2}}$	$-\frac{[3]^2D_0}{D_2D_{-2}}\sqrt{\frac{D_3D_{-3}}{D_1D_{-1}}}$	$\bar{S}_{[4,7]}$	$\frac{D_2}{[2]D_0}\sqrt{\frac{D_{-3}}{D_{-1}}}$	$-\frac{\sqrt{D_{-1}D_{-3}D_2}}{[2]D_0\sqrt{D_{-2}}}$	$\frac{\sqrt{D_{-1}D_{-3}D_2}}{[2]D_0\sqrt{D_{-2}}}$
$\frac{\sqrt{D_3D_{-3}}}{D_0}$	0	0	$\frac{[3]\sqrt{D_1D_3}}{D_0D_2D_{-2}}$	$-\frac{[3]}{D_{-2}D_0D_2}$	$\frac{[3]\sqrt{D_{-1}D_{-3}}}{D_{-2}D_0D_2}$	$-\frac{[3]\sqrt{D_{-1}D_1D_{-3}D_3}}{D_{-2}D_0D_2}$	0	0	0
$\frac{[3]}{[2]D_1}\sqrt{\frac{D_3}{D_{-1}}}$	$-\frac{[3]}{[2]^2D_0}\sqrt{\frac{D_3D_{-2}}{D_2D_1}}$	$-\frac{[3]}{[2]^2D_0}\sqrt{\frac{D_3D_{-2}}{D_1D_2}}$	$-\frac{[3]^2D_0}{D_2D_{-2}}\sqrt{\frac{D_3D_{-3}}{D_1D_{-1}}}$	$\frac{[3]\sqrt{D_{-1}D_{-3}}}{D_{-2}D_0D_2}$	$\bar{S}_{[6,6]}$	$\bar{S}_{[6,7]}$	$\frac{D_{-2}}{[2]D_0}\sqrt{\frac{D_3}{D_1}}$	$\frac{\sqrt{D_1D_3D_{-2}}}{[2]D_0\sqrt{D_2}}$	$-\frac{\sqrt{D_1D_3D_{-2}}}{[2]D_0\sqrt{D_2}}$
$\frac{[3]}{D_0}\frac{1}{\sqrt{D_1D_{-1}}}$	$\bar{S}_{[2,7]}$	$\bar{S}_{[2,7]}$	$\bar{S}_{[4,7]}$	$-\frac{[3]\sqrt{D_{-1}D_1D_{-3}D_3}}{D_{-2}D_0D_2}$	$\bar{S}_{[6,7]}$	$\bar{S}_{[7,7]}$	$\frac{[4]}{[2]D_0}$	$-\frac{\{A^2\}}{\{q\}}\frac{1}{D_0^2\sqrt{D_2D_{-2}}}$	$\frac{\{A^2\}}{\{q\}}\frac{1}{D_0^2\sqrt{D_2D_{-2}}}$
$\frac{[3]}{D_0}\frac{1}{\sqrt{D_1D_{-1}}}$	$-\frac{\sqrt{D_2D_{-2}}}{[2]D_0}$	$-\frac{\sqrt{D_2D_{-2}}}{[2]D_0}$	$\frac{D_2}{[2]D_0}\sqrt{\frac{D_{-3}}{D_{-1}}}$	0	$\frac{D_{-2}}{[2]D_0}\sqrt{\frac{D_3}{D_1}}$	$\frac{[4]}{[2]D_0}$	$-\frac{1}{D_0}$	0	0
0	$-\frac{D_1}{[2]D_0}$	$\frac{D_{-1}}{[2]D_0}$	$-\frac{\sqrt{D_{-1}D_{-3}D_2}}{[2]D_0\sqrt{D_{-2}}}$	0	$\frac{\sqrt{D_1D_3D_{-2}}}{[2]D_0\sqrt{D_2}}$	$-\frac{\{A^2\}}{\{q\}}\frac{1}{D_0^2\sqrt{D_2D_{-2}}}$	0	$-\frac{1}{D_0}$	$\frac{1}{D_0}$
0	$-\frac{D_{-1}}{[2]D_0}$	$\frac{D_1}{[2]D_0}$	$\frac{\sqrt{D_{-1}D_{-3}D_2}}{[2]D_0\sqrt{D_{-2}}}$	0	$-\frac{\sqrt{D_1D_3D_{-2}}}{[2]D_0\sqrt{D_2}}$	$\frac{\{A^2\}}{\{q\}}\frac{1}{D_0^2\sqrt{D_2D_{-2}}}$	0	$\frac{1}{D_0}$	$-\frac{1}{D_0}$

$$\bar{S}_{[2,7]} = -\frac{[2][3]^2D_{-2}D_0D_2 - [2]D_2^2D_0D_{-2}^2 - [3]^3D_0(D_2D_{-3} + D_3D_{-2}) + [3]D_2D_{-2}(D_3 + D_{-3})}{[2]^2D_0^2D_1D_{-1}\sqrt{D_2D_{-2}}}$$

$$\bar{S}_{[4,4]} = \frac{[2]^2(D_1 + D_{-1}) - D_{-3}D_0D_4}{[2]^2D_0D_{-1}D_{-2}D_2}$$

$$\bar{S}_{[4,7]} = \sqrt{\frac{D_{-3}}{D_{-1}^3}} \frac{[2]^2[3]^2 + 2[3]D_2D_{-2}(D_{-1}^2 + D_1^2) - [3]^2D_0D_1(D_2D_{-3} + D_3D_{-2}) - [2]D_{-1}D_{-2}^2D_2^3}{[2]^2D_0D_2D_{-2}D_1^2}$$

$$\bar{S}_{[6,7]} = \sqrt{\frac{D_3}{D_1^3}} \frac{[2]^2[3]^2 + 2[3]D_2D_{-2}(D_{-1}^2 + D_1^2) - [3]^2D_0D_{-1}(D_2D_{-3} + D_3D_{-2}) - [2]D_1D_2^2D_{-2}^3}{[2]^2D_0D_2D_{-2}D_{-1}^2}$$

$$\bar{S}_{[6,6]} = \frac{[2]^2(D_1 + D_{-1}) - D_3D_0D_{-4}}{[2]^2D_0D_1D_2D_{-2}}$$

$$\bar{S}_{[7,7]} = \frac{[2]D_{-2}D_2(D_2D_{-1}^2 + D_{-2}D_1^2) - 2[3](D_{-1}^3 + D_1^3)}{[2]D_0^2D_{-1}^2D_1^2} + \frac{[2][3]^2\{A^2\}^2}{\{q\}^2D_{-2}D_2D_0^3D_1^2D_{-1}^2(D_1D_{-2} + D_{-1}D_2)} + \frac{[3]^2(D_2D_{-3} + D_3D_{-2})}{D_{-2}D_0D_2(D_1D_{-2} + D_{-1}D_2)}$$

(74)

$\sqrt{\frac{D_{-2}}{[2]^2[4]D_1}}$	$\sqrt{\frac{[3]D_3D_2D_{-2}}{[2]^2[6]D_1D_0D_{-1}}}$	$\sqrt{\frac{D_2}{[2]^2[4]D_{-1}}}$	$\sqrt{\frac{[3]^2D_{-2}D_{-3}}{[2]^2[4][5]D_1D_{-1}}}$	$\sqrt{\frac{[3]D_2D_{-2}D_{-3}}{[2]^2[6]D_1D_0D_{-1}}}$	$\sqrt{\frac{[3]^2D_3D_2}{[2]^2[4][5]D_1D_{-1}}}$	$\sqrt{\frac{D_2D_{-2}}{[5]D_1D_{-1}}}$	$\sqrt{\frac{D_2D_{-2}}{[5]D_1D_{-1}}}$	0	0
$\frac{[3]\sqrt{D_{-1}D_2}}{[2]^2\sqrt{[4]D_0}}$	$\frac{\sqrt{[3]D_3}}{[2]^2\sqrt{[6]D_0}}$	$\frac{[3]\sqrt{D_1D_{-2}}}{[2]^2D_0\sqrt{[4]}}$	$-\frac{[3]\sqrt{D_2D_{-3}}}{[2]^2D_0\sqrt{[4][5]}}$	$\frac{\sqrt{[3]D_{-3}}}{[2]^2\sqrt{D_0[6]}}$	$-\frac{[3]\sqrt{D_3D_{-2}}}{[2]^2D_0\sqrt{[4][5]}}$	$-\frac{[4]D_0+[3](D_2-D_{-2})}{2[2]^2\sqrt{[5]D_0}}$	$-\frac{[4]D_0-[3](D_2-D_{-2})}{2[2]^2\sqrt{[5]D_0}}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{[3]\sqrt{D_{-1}D_2}}{[2]^2\sqrt{[4]D_0}}$	$\frac{\sqrt{[3]D_3}}{[2]^2\sqrt{[6]D_0}}$	$\frac{[3]\sqrt{D_1D_{-2}}}{[2]^2D_0\sqrt{[4]}}$	$-\frac{[3]\sqrt{D_2D_{-3}}}{[2]^2D_0\sqrt{[4][5]}}$	$\frac{\sqrt{[3]D_{-3}}}{[2]^2\sqrt{D_0[6]}}$	$-\frac{[3]\sqrt{D_3D_{-2}}}{[2]^2D_0\sqrt{[4][5]}}$	$-\frac{[4]D_0+[3](D_2-D_{-2})}{2[2]^2\sqrt{[5]D_0}}$	$-\frac{[4]D_0-[3](D_2-D_{-2})}{2[2]^2\sqrt{[5]D_0}}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{\sqrt{D_{-3}}}{[2]^2\sqrt{[4]}\sqrt{D_{-2}}}$	$\frac{\sqrt{D_0D_3D_{-3}[3]^3}}{[2]^2\sqrt{[6]D_{-1}D_{-2}D_2}}$	$-\frac{[3]\sqrt{D_1D_{-3}}}{[2]^2\sqrt{[4]D_2D_{-1}}}$	$-\frac{[4]D_{-1}+[2][3]D_1}{[2]^2\sqrt{[4][5]D_{-1}D_{-2}}}$	$S_{[4,5]}$	$-\frac{[3]^2D_3D_{-3}}{[2]^2\sqrt{[4][5]D_2D_{-1}}}$	$\frac{\sqrt{D_{-3}(D_2+D_{-1})}}{[2]\sqrt{[5]D_{-1}D_{-2}D_2}}$	$\frac{\sqrt{D_{-3}(D_2-D_{-1})}}{[2]\sqrt{[5]D_{-1}D_{-2}D_2}}$	0	0
$\frac{\sqrt{D_1D_3D_{-3}}}{[2]D_0\sqrt{[4]D_{-2}}}$	$-\frac{\sqrt{[3]D_{-3}D_{-1}D_1}}{[2]\sqrt{[6]D_0D_2D_{-2}}}$	$-\frac{\sqrt{D_{-3}D_{-1}D_3}}{[2]D_0\sqrt{[4]D_2}}$	$-\frac{\sqrt{[3]D_3D_1D_{-1}}}{[2]D_0\sqrt{[4][5]D_{-2}}}$	$\frac{\sqrt{[3]D_1D_{-1}D_3}}{[2]\sqrt{[6]D_0D_2D_{-2}}}$	$-\frac{\sqrt{[3]D_1D_{-1}D_{-3}}}{[2]D_0\sqrt{[4][5]D_2}}$	$\frac{\sqrt{D_1D_{-1}D_3D_{-3}}}{D_0\sqrt{[5]D_2D_{-2}}}$	$\frac{\sqrt{D_1D_{-1}D_3D_{-3}}}{D_0\sqrt{[5]D_2D_{-2}}}$	0	0
$\frac{[3]\sqrt{D_{-1}D_3}}{[2]^2\sqrt{[4]D_1D_{-2}}}$	$S_{[6,2]}$	$-\frac{\sqrt{D_3}}{[2]^2\sqrt{[4]D_2}}$	$\frac{[3]^2\sqrt{D_3D_{-3}}}{[2]^2\sqrt{[4][5]D_1D_{-2}}}$	$-\frac{\sqrt{D_0D_3D_{-3}[3]^3}}{[2]^2\sqrt{[6]D_1D_2D_{-2}}}$	$\frac{[2][3]D_{-1}+[4]D_1}{[2]^3\sqrt{[4][5]D_1D_2}}$	$-\frac{\sqrt{D_3(D_{-2}-D_1)}}{[2]\sqrt{[5]D_1D_2D_{-2}}}$	$-\frac{\sqrt{D_3(D_{-2}+D_1)}}{[2]\sqrt{[5]D_1D_2D_{-2}}}$	0	0
$-\frac{\sqrt{D_{-1}^3}}{[2]D_0\sqrt{[4]D_{-2}}}$	$S_{[7,2]}$	$\frac{\sqrt{D_1^3}}{[2]D_0\sqrt{[4]D_2}}$	$-\frac{\sqrt{D_{-3}([2][3]D_1+[4]D_{-1})}}{[2]^2D_0\sqrt{[4][5]D_{-2}}}$	$S_{[7,5]}$	$\frac{\sqrt{D_3}([2][3]D_{-1}+[4]D_1)}{[2]^2D_0\sqrt{[4][5]D_2}}$	$-\frac{\frac{\{A^2\}}{\{q\}}+1}{D_0\sqrt{[5]D_2D_{-2}}}$	$-\frac{\frac{\{A^2\}}{\{q\}}-1}{D_0\sqrt{[5]D_2D_{-2}}}$	0	0
$-\frac{\sqrt{D_{-1}D_{-2}}}{D_0\sqrt{[4]}}$	0	$\frac{\sqrt{D_1D_2}}{D_0\sqrt{[4]}}$	$\frac{\sqrt{D_{-3}D_{-2}}}{D_0\sqrt{[4][5]}}$	0	$-\frac{\sqrt{D_3D_2}}{D_0\sqrt{[4][5]}}$	$\frac{\sqrt{D_2D_{-2}}}{D_0\sqrt{[5]}}$	$-\frac{\sqrt{D_2D_{-2}}}{D_0\sqrt{[5]}}$	0	0
$\frac{\sqrt{D_{-1}D_2}}{[2]D_0\sqrt{[4]}}$	$-\frac{\sqrt{[3]D_3}}{[2]\sqrt{[6]D_0}}$	$\frac{\sqrt{D_1D_{-2}}}{[2]D_0\sqrt{[4]}}$	$-\frac{\sqrt{D_{-3}D_2}}{[2]D_0\sqrt{[4][5]}}$	$-\frac{\sqrt{[3]D_{-3}}}{[2]\sqrt{[6]D_0}}$	$-\frac{\sqrt{D_3D_{-2}}}{[2]D_0\sqrt{[4][5]}}$	$\frac{[6]D_0+[3](D_{-2}-D_2)}{2[2][3]D_0\sqrt{[5]}}$	$\frac{[6]D_0+[3](D_2-D_{-2})}{2[2][3]D_0\sqrt{[5]}}$	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{\sqrt{D_{-1}D_2}}{[2]D_0\sqrt{[4]}}$	$-\frac{\sqrt{[3]D_3}}{[2]\sqrt{[6]D_0}}$	$\frac{\sqrt{D_1D_{-2}}}{[2]D_0\sqrt{[4]}}$	$-\frac{\sqrt{D_{-3}D_2}}{[2]D_0\sqrt{[4][5]}}$	$-\frac{\sqrt{[3]D_{-3}}}{[2]\sqrt{[6]D_0}}$	$-\frac{\sqrt{D_3D_{-2}}}{[2]D_0\sqrt{[4][5]}}$	$\frac{[6]D_0+[3](D_{-2}-D_2)}{2[2][3]D_0\sqrt{[5]}}$	$\frac{[6]D_0+[3](D_2-D_{-2})}{2[2][3]D_0\sqrt{[5]}}$	$-\frac{1}{2}$	$\frac{1}{2}$

$$\begin{aligned}
S_{[4,5]} &= \frac{[2][3]D_{-1}D_{-2} - [6]D_{-1}D_2 - [3][4]D_1D_2}{[2]^2[4]\sqrt{[3][6]D_0D_{-1}D_{-2}D_2}} \\
S_{[6,2]} &= -\frac{[2][3]D_1D_2 - [6]D_1D_{-2} - [3][4]D_{-1}D_{-2}}{[2]^2[4]\sqrt{[3][6]D_0D_1D_{-2}D_2}} \\
S_{[7,2]} &= -\sqrt{D_3} \frac{[2][3]D_1D_2 - [6]D_1D_{-2} - [3][4]D_{-1}D_{-2}}{[2][4]\sqrt{[3][6]D_0^3D_{-2}D_2}} \\
S_{[7,5]} &= \sqrt{D_{-3}} \frac{[2][3]D_{-1}D_{-2} - [6]D_{-1}D_2 - [3][4]D_1D_2}{[2][4]\sqrt{[3][6]D_0^3D_{-2}D_2}}
\end{aligned} \tag{75}$$